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A VARIATIONAL FRAMEWORK FOR NONLINEAR VISCOELASTIC AND VISCOPLASTIC MODELS IN FINITE DEFORMATION REGIME

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Abstract. *The goal of this work is to provide a general framework for constitutive viscoelastic and viscoplastic models based on the theoretical background proposed in Ortiz and Stainier, Comput. Meth. App. Mech. Engng., Vol. 171, 419–444 (1999). Thus, the approach is qualified as variational since the constitutive updates obey a minimum principle within each load increment. The set of internal variables is strain-based and thus employs, according to the specific model chosen, multiplicative decomposition of strain in elastic and irreversible components. Inserted in the same theoretical framework, the present approach for viscoelasticity shares the same technical procedures used for analogous models of plasticity or viscoplasticity, say, the solution of a minimization problem to identify inelastic updates and the use of exponential mapping for time integration. Spectral decomposition is explored in order to accommodate, into analytically tractable expressions, a wide set of specific models. Moreover, it is shown that, through appropriate choices of the constitutive potentials, the proposed formulation is able to reproduce results obtained elsewhere in the literature. Finally, different numerical examples are included to show the characteristics of the present approach and to compare results with others found in literature when possible.*

1 INTRODUCTION

Many different models for viscoelastic materials in finite deformation regime are found in literature. However, in contrast with what we see in small deformation models, the choice of convenient internal variables and evolution laws is not trivial nor unique, leading to different formulations. From one side, we recall the the work of Simo[1] in which an additive decomposition of stresses in equilibrium and non equilibrium contributions is stated and, more relevant, the evolution law is defined as a linear differential equation on the non-equilibrium stresses. This approach was later followed, among many others, by Holzapfel and Simo,[2] Holzapfel[3] or, more recently, Holzapfel and Gasser,[4] and Bonet.[5] Multiplicative decomposition of strains applied to viscoelastic constitutive equations goes back to the work of Sidoroff[6] and later to several others.[7–9] In particular Reese and Govindjee[10] discussed the ability of different models to reproduce nonlinear viscous behavior and they proposed a model which is not restricted to small perturbations away from thermodynamic equilibrium.

The goal of this work is to provide a general framework for constitutive viscoelastic models based on the mathematical background proposed in Ortiz and Stainier,[11] and Stainier.[12] The approach is qualified as variational since the constitutive updates obey a minimum principle within each load increment. The set of internal variables is strain-based and thus employs, according to the specific model chosen, multiplicative decomposition of strain in elastic and viscous components.

Inserted in the same theoretical framework, this particularization and that for plasticity or viscoplasticity, share the same technical procedures to deal with the local nonlinear constitutive problem, i.e. the solution of a minimization problem to identify inelastic updates and the use of exponential mapping for time integration.[13,14] However, instead of using the classic decomposition of inelastic strains into “size” and “direction”, we take profit of a spectral decomposition that provides additional facilities to accommodate, into simple analytical expressions, a wide set of viscous models. It is also possible to show that an appropriate choice of constitutive potentials allows to retrieve other models in literature.

2 VARIATIONAL FORM OF CONSTITUTIVE EQUATIONS

Using conventional notation, let us call $\mathbf{F} = \nabla_0 \mathbf{x}$ the gradient of deformations, and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ the Cauchy strain tensor, respectively. These values may be decomposed in volumetric and isochoric parts. The isochoric tensors are defined as follows:

$$\hat{\mathbf{F}} = \frac{1}{J^{1/3}}\mathbf{F}, \quad J = \det(\mathbf{F}), \quad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}} = \frac{1}{J^{2/3}}\mathbf{F}^T\mathbf{F}, \tag{1}$$

We will work in the framework of irreversible thermodynamics, with internal variables. Thus, we define a general set $\mathcal{E} = \{\mathbf{F}, \mathbf{F}^i, \mathbf{Q}\}$ of external and internal variables, where $\mathbf{F}^i$ is the inelastic part of the (total) deformation, and $\mathbf{Q}$ contains all the remaining internal variables of the model. In addition, a multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^i$ of the gradient of deformations is considered. We assume the existence of a free energy potential $W(\mathcal{E})$ and a dissipative potential

$\phi(\dot{\mathbf{F}};\mathcal{E})$, such that the Piola-Kirchhoff stress tensor, comprised of an equilibrium (elastic) and a dissipative (viscous) components, is derived as follows:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}(\mathcal{E}) + \frac{\partial \phi}{\partial \dot{\mathbf{F}}}(\dot{\mathbf{F}};\mathcal{E}). \tag{2}$$

In addition, another dissipative potential $\psi(\dot{\mathbf{F}}^i,\dot{\mathbf{Q}};\mathcal{E})$ is included to characterize the irreversible behavior related to the inelastic tensor $\mathbf{F}^i$, such that

$$\mathbf{T} = -\frac{\partial W}{\partial \mathbf{F}^i}(\mathcal{E}) = \frac{\partial \psi}{\partial \dot{\mathbf{F}}^i}(\dot{\mathbf{F}}^i,\dot{\mathbf{Q}};\mathcal{E}), \qquad \mathbf{A} = -\frac{\partial W}{\partial \mathbf{Q}}(\mathcal{E}) = \frac{\partial \psi}{\partial \dot{\mathbf{Q}}^i}(\dot{\mathbf{F}}^i,\dot{\mathbf{Q}};\mathcal{E}). \tag{3}$$

It was shown[11,12] that an incremental version of the above equations, constituting an incremental update method for the material state, can be obtained from the following incremental potential:

$$\mathcal{W}(\mathbf{F}_{n+1};\mathcal{E}_n) = \Delta t \phi\left(\mathring{\mathbf{F}},\mathcal{E}_n\right) + \min_{\substack{\mathbf{F}^i_{n+1}\\ \mathbf{Q}_{n+1}}} \left\{ W(\mathcal{E}_{n+1}) - W(\mathcal{E}_n) + \Delta t \psi\left(\mathring{\mathbf{F}}^i,\mathring{\mathbf{Q}};\mathcal{E}_n\right)\right\}, \tag{4}$$

where $\mathring{\mathbf{F}}\left(\mathbf{F}_{n+1},\mathcal{E}_n\right)$, $\mathring{\mathbf{F}}^i\left(\mathbf{F}^i_{n+1},\mathcal{E}_n\right)$ and $\mathring{\mathbf{Q}}\left(\mathbf{Q}_{n+1},\mathcal{E}_n\right)$ are suitable incremental approximations of the rate variables $\dot{\mathbf{F}}, \dot{\mathbf{F}}^i$ and $\dot{\mathbf{Q}}$ respectively.

3 VISCOPLASTICITY MODELS

We start by recalling the variational formulation of visoplasticity, as proposed in previous work.[11,12] This will provide a background against which differences and similarities between viscoplasticity and viscoelasticity can be contrasted.

Viscoplastic solids are characterized by the existence of a certain class of deformations $\mathbf{F}^p$, or *plastic* deformations, which leave the crystal lattice undistorted and unrotated, and, consequently, induce no long-range stresses. In addition to the plastic deformation $\mathbf{F}^p$, some degree of lattice distortion $\mathbf{F}^e$, or *elastic* deformation, may also be expected in general. One therefore has, locally,

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{5}$$

This multiplicative elastic-plastic kinematics was first suggested by Lee,[15] and further developped and used by many others.

We postulate the existence of a Helmholtz free energy density of the form

$$W = W(\mathbf{F}^e,\mathbf{F}^p,\mathbf{Q}) = W(\mathbf{F}\mathbf{F}^{p-1},\mathbf{F}^p,\mathbf{Q}) \equiv W(\mathbf{F},\mathbf{F}^p,\mathbf{Q}) \tag{6}$$

where $\mathbf{Q} \in \mathbb{R}^N$ denotes some suitable collection of internal variables. The complete set of internal variables is, therefore $\{\mathbf{F}^p,\mathbf{Q}\}$. The *local state* of the material is then described by the variables $\{\mathbf{F},\mathbf{F}^p,\mathbf{Q}\}$. In materials such as metals, the elastic response is ostensibly independent of the internal processes and the free energy (6) decomposes additively as

$$W = W^e(\mathbf{F}\mathbf{F}^{p-1}) + W^p(\mathbf{F}^p,\mathbf{Q}) \tag{7}$$

The function W^e determines the elastic response of the metal, e. g., upon unloading, whereas the function W^p describes the hardening of the material. Physically, W^p represents the stored energy due to the plastic working of the material. In the case where the state of hardening if fully described by a single internal variable, i. e., $N = 1$, $\mathbf{Q} = \{\bar{\epsilon}^p\}$, the material is said to undergo isotropic hardening.

A possible extension of von Mises' theory of plasticity to the finite deformation range is obtained by postulating a flow rule of the form:

$$\dot{\mathbf{F}}^p \mathbf{F}^{p-1} = \dot{\bar{\epsilon}}^p \mathbf{M} \tag{8}$$

where $\bar{\epsilon}^p$ is the effective plastic strain and the symmetric matrix $\mathbf{M}$ is required to satisfy the incompressibility constraint:

$$\text{trace}(\mathbf{M}) = 0 \tag{9}$$

and the normalization condition:

$$\mathbf{M} \cdot \mathbf{M} = \frac{3}{2} \tag{10}$$

but is otherwise unspecified. The normalization condition (10) is chosen so that $\dot{\bar{\epsilon}}^p$ coincides with the rate of axial stretching in the uniaxial test. The flow rule (8) is a constraint relating the internal variables $\mathbf{F}^p$ and $\bar{\epsilon}^p$, which must be accounted for in the minimization (4) (the minimization is then also carried over $\mathbf{M}$).

Plastic deformations are incrementally updated by the exponential mapping

$$\mathbf{F}^p_{n+1} = \exp\left[\Delta\bar{\epsilon}^p \mathbf{M}\right] \mathbf{F}^p_n \tag{11}$$

The exponential mapping has the convenient property of preserving the isochoric character of plastic deformations.[13,14]

4 A GROUP OF VISCO-HYPERELASTIC MODELS

4.1 General formulation

A quite general group of viscoelastic materials can be modelled within the present variational framework. Due to the possibility of obtaining analytical or semi-analytical expression for the constitutive updates, only isotropic models will be considered now. However, no theoretical constraints to include more general behaviors are found. The rheological mechanism shown in Figure 1 is taken as a basis to include different potentials expressions in (4). The model is based on the following assumptions:

- The elastic part of the Kelvin branch is split in isochoric and volumetric energies. The isochoric part is an isotropic function of $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}}$:

$$\varphi(\hat{\mathbf{C}}) = \varphi(c_1, c_2, c_3), \tag{12}$$

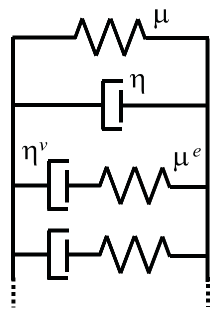


Figure 1: Generalized Kelvin-Maxwell model.

where c_j are the eigenvalues of $\hat{\mathbf{C}}$. The volumetric part may be defined using the usual expression $U(J) = \frac{K}{2}[\ln J]^2$, The viscous part of the Kelvin branch is an isotropic function of the symmetric part of the rate of deformation:

$$\phi(\mathbf{D}) = \phi(d_1, d_2, d_3) \quad \text{with} \quad \mathbf{D} = \text{dev}\left(\text{sym}\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)\right), \tag{13}$$

where d_j are the eigenvalues of $\mathbf{D}$.

- The Maxwell branch, connected in parallel, is based on a multiplicative split of strains in an elastic and an isochoric inelastic (viscous) part:

$$\hat{\mathbf{F}} = \hat{\mathbf{F}}^e\mathbf{F}^v \Longrightarrow \hat{\mathbf{F}}^e = \hat{\mathbf{F}}\mathbf{F}^{v-1}, \quad \det \mathbf{F}^v = 1. \tag{14}$$

A flow rule for the internal variable $\mathbf{F}^v$ can be written as:

$$\dot{\mathbf{F}}^v = \mathbf{D}^v\mathbf{F}^v = \left(d_j^v \mathbf{M}_j^v\right)\mathbf{F}^v, \tag{15}$$

in which the spectral decomposition of $\mathbf{D}^v = \text{sym}(\dot{\mathbf{F}}^v\mathbf{F}^{v-1})$ in eigenvalues d_j^v and eigenprojections $\mathbf{M}_j^v,\ j = 1,2,3,$ was used. The scalars d_j are chosen to be the internal variables contained in the set $\dot{\mathbf{Q}} = \{d_1, d_2, d_3\}$. In this case, it is important to note that (15) is a constraint relating the internal variables $\mathbf{F}^v$ and $\mathbf{Q}$. The elastic and viscous potentials associated to this branch are assumed to be isotropic functions of the elastic deformation and viscous stretching, and thus depend on their eigenvalues:

$$\varphi^e(\hat{\mathbf{C}}^e) = \varphi^e(c_1^e, c_2^e, c_3^e) \quad \text{and} \quad \psi(\mathbf{D}^v) = \psi(d_1^v, d_2^v, d_3^v), \tag{16}$$

where c_j^e are the eigenvalues of $\hat{\mathbf{C}}^e$.

- Viscous deformations are incrementally updated by exponential mappings:

$$\Delta\hat{\mathbf{F}} = \hat{\mathbf{F}}_{n+1}\hat{\mathbf{F}}_n^{-1} = \exp[\Delta t\mathbf{D}] \quad \Rightarrow \mathbf{D} = \frac{\Delta q_j}{\Delta t}\mathbf{M}_j = \frac{1}{2\Delta t}\ln\left(\Delta\hat{\mathbf{C}}\right). \tag{17}$$

$$\Delta \mathbf{F}^v = \mathbf{F}^v_{n+1}\mathbf{F}^{v-1}_n = \exp[\Delta t \mathbf{D}^v] \quad \Rightarrow \mathbf{D}^v = \frac{\Delta q^v_j}{\Delta t}\mathbf{M}^v_j = \frac{1}{2\Delta t}\ln\left(\Delta \mathbf{C}^v\right). \tag{18}$$

Expressions (17) and (18) show that $\mathbf{D}$ and $\mathbf{D}^v$ are approximated by incremental expressions of $\Delta\hat{\mathbf{C}}$ and $\Delta\mathbf{C}^v$ respectively. The exponential mapping has the particular convenient property of providing an isochoric tensor for any traceless argument.[13,14]

Taking into account (17) and (18), the minimizing variables $\mathbf{Q}_{n+1}, \mathbf{F}^v_{n+1}$ in (4) are replaced by the new incremental variables $\Delta q^v_j, \mathbf{M}^v_j$.:

$$\begin{aligned}\mathcal{W}(\mathbf{F}_{n+1};\mathcal{E}_n) = \mathcal{W}(\mathbf{C}_{n+1};\mathcal{E}_n) &= \Delta\varphi(\hat{\mathbf{C}}_{n+1}) + \Delta t \phi\left(\tfrac{\Delta q^v_j}{\Delta t}\right) + \Delta U(\theta_{n+1}) \\ &+ \min_{\mathbf{M}^v_j, \Delta q^v_j}\left\{\Delta\varphi^e(\hat{\mathbf{C}}^e_{n+1}) + \Delta t\psi\left(\tfrac{\Delta q^v_j}{\Delta t}\right)\right\},\end{aligned} \tag{19}$$

such that

$$\Delta q^v_j \in K_Q = \{p_j \in \mathbb{R} : p_1 + p_2 + p_3 = 0\}, \tag{20}$$

$$\mathbf{M}^v_j \in K_M = \{\mathbf{N}_j \in Sym : \mathbf{N}_j \cdot \mathbf{N}_j = 1,\ \ \mathbf{N}_i \cdot \mathbf{N}_j = 0,\ \ i \neq j\}. \tag{21}$$

The set K_Q enforces the traceless form of $\mathbf{D}^v$, while the set K_M accounts for usual properties of eigenprojections. Moreover, it is easy to verify that both sets are convex on their respective variables. Given isotropic expressions for energy functions, the minimization in (19) can be performed analytically. A simple extension to this model can be obtained by considering a set of P Maxwell branches, as seen in Figure 1. More details on the practical implementation of this formulation, including the derivation of a consistent tangent operator can be found elsewhere.[16]

4.2 Hencky and Ogden models

Hencky models are based on quadratic forms of logarithmic strain tensors:

$$\varphi = \mu \sum_{j=1}^{3} (\epsilon_j)^2, \quad \phi = \eta \sum_{j=1}^{3} (d_j)^2, \tag{22}$$

$$\varphi^e = \mu^e \sum_{j=1}^{3}\left(\epsilon^e_j\right)^2, \quad \psi = \eta^v \sum_{j=1}^{3}\left(d^v_j\right)^2. \tag{23}$$

In this case, it is particularly convenient to obtain simple uncoupled linear expressions for the minimizing argument Δq^v_j. In spite of the facility offered by Hencky models in terms of analytical treatment, it is well known that these type of hyperelastic potentials do not fit well the behavior of rubber-like materials. For that case, a more adequate choice may be the Ogden model which has also the capability of generalizing other models like neo-Hookean and Mooney-Rivlin. Ogden models are based on the following potentials:

$$\varphi = \sum_{j=1}^{3}\sum_{p=1}^{N}\frac{\mu_p}{\alpha_p}\left([\exp(\epsilon_j)]^{\alpha_p} - 1\right), \quad \phi = \sum_{j=1}^{3}\sum_{p=1}^{N}\frac{\eta_p}{\alpha_p}\left([\exp(d_j)]^{\alpha_p} - 1\right), \tag{24}$$

$$\varphi^e = \sum_{j=1}^{3}\sum_{p=1}^{N}\frac{\mu^e_p}{\alpha_p}\left(\left[\exp(\epsilon^e_j)\right]^{\alpha_p} - 1\right), \quad \psi = \sum_{j=1}^{3}\sum_{p=1}^{N}\frac{\eta^v_p}{\alpha_p}\left(\left[\exp(d^v_j)\right]^{\alpha_p} - 1\right). \tag{25}$$

5 NUMERICAL EXAMPLES

5.1 Shear test

This example presents a pure shear tests of a single 3D element (Figure 2). Material parameters and load characteristics were taken from an equivalent example in Reese and Govindjee,[10] in order to perform some useful comparisons. The lateral displacement u_x follows a sinusoidal law

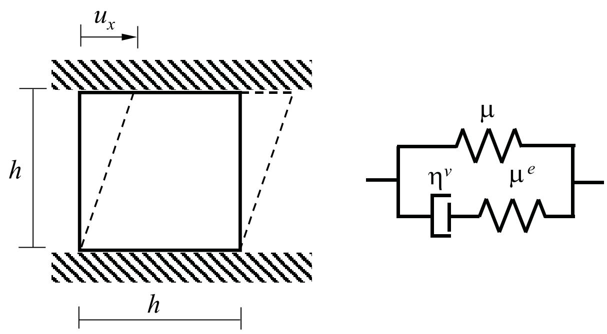


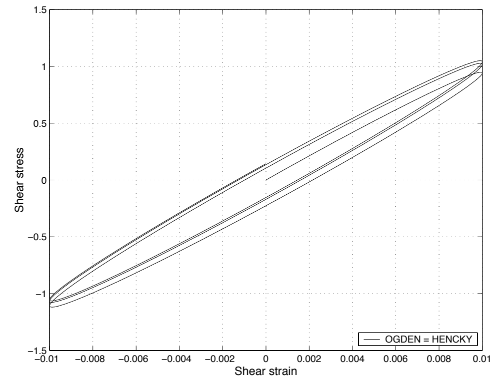
Figure 2: Cyclic shear test

$u_x = U \sin wt$, where $w = 0.3\ s^{-1}$. The material was assumed to be almost incompressible through the choice of a high value for the bulk modulus K. Two different models for φ were used: Ogden model and Hencky model. In the case of Ogden, we used the following six-parameter fitting: $\mu_1 = 20$, $\mu_2 = -7$, $\mu_3 = 1.5$, $\alpha_1 = 1.8$, $\alpha_2 = -2$, $\alpha_3 = 7$. For the Hencky model, the value $\mu = \sum_i \frac{1}{2}\mu_i\alpha_i = 30.25$ was used, which is the consistent equivalent shear modulus for small deformations. The Maxwell branch uses Hencky model for both potentials with $\mu^e = 77.77$ and viscous coefficient η^v such that $\tau = \frac{\eta^v}{\mu^e} = 17.5$.

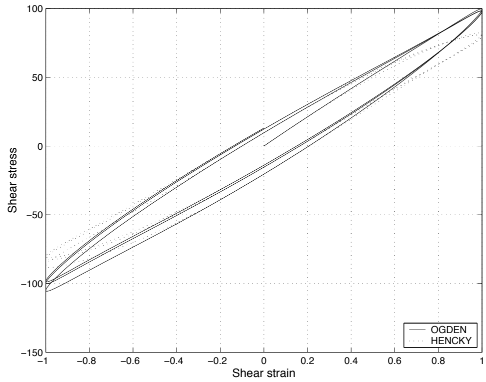
The time evolution of Cauchy stresses σ_{xy} as a function of shear strain $\mathbf{C}_{xy}$, for different shear amplitudes, is shown in Figure 3. In the case of small strains both models (Ogden or Hencky main spring) give identical results, and match quite well equivalent results in the reference work.[10] As expected, the behavior of the main spring is determinant on the behavior of the whole system for deformations higher than unity. Comparing the results of the Ogden-based model with those of Reese and Govindjee[10] it is possible to see a close correlation of maximum values of stress for all four cases. However, hysteresis loops clearly look “thinner” as the deformation grows along the cycle. This behavior is in agreement with the fact that the Hencky model used in the Maxwell branch provides a contribution in stress much more lower than a corresponding Ogden model for high deformations.

5.2 Viscoelastic support

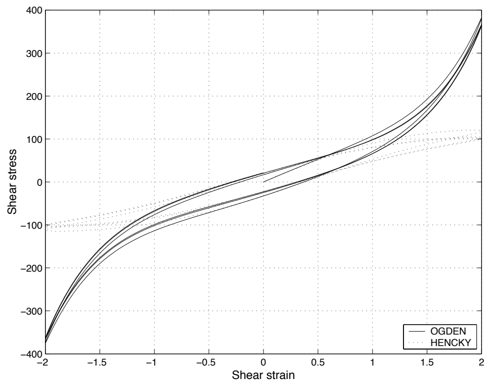
This example simulates a viscoelastic cylindrical support clamped on its exterior surface and with a rigid edge connected to a vibrating device (see Figure 4).



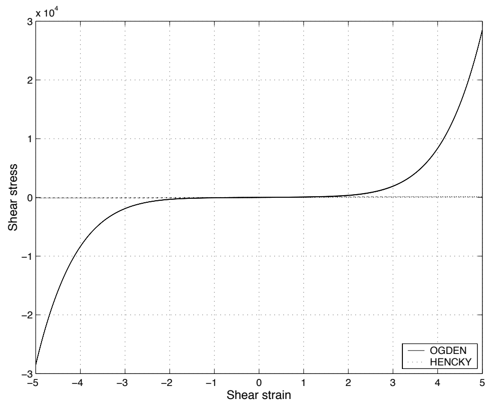
(a) Shear amplitude: 0.01



(b) Shear amplitude: 1



(c) Shear amplitude: 2



(d) Shear amplitude: 5

Figure 3: Stress-strain curves in cyclic shear test

The harmonic vibration $u = U \sin wt$, with $w = 10\ s^{-1}$ occur along different directions. The first one is a lateral translation with maximum displacement $U_x = \frac{3}{8}(R_e - R_i)$. The second one is composed by a axial translation ($U_y = R_i$) and rotation ($\theta_y = 45°$). Dimensions of the viscoelastic support are $R_i = 15mm$, $R_e = 30mm$ and $L = 50mm$. Figure 5(a) shows the maximum amplitude of deformations for a lateral motion while Figure 6(a) illustrates the case of axial (rotation plus translation) motion. The isochoric nature of the material shows up in Figure 5(a) where the compressed and tractioned sides of the cylinder deform accordingly to preserve this constraint. Two material models were run, both sharing the same constitutive characteristics in the small strain regime.

- Case 1: Ogden model for both springs potentials φ and φ^e ($\mu_i^e = 4\mu_i$) : $\mu_1 = 3.5$; $\mu_2 = 0.011$; $\mu_3 = 0.0015$ and $\alpha_1 = 0.1$; $\alpha_2 = 2.0$; $\alpha_3 = 9$ ($\mu_{eq} = \sum_i \frac{1}{2}\mu_i\alpha_i = 0.19275$ MPa). The viscous potential ψ is of Hencky type with $\eta^v = 7.69$ ($\tau = 0.4$ s). Potential ϕ is null.

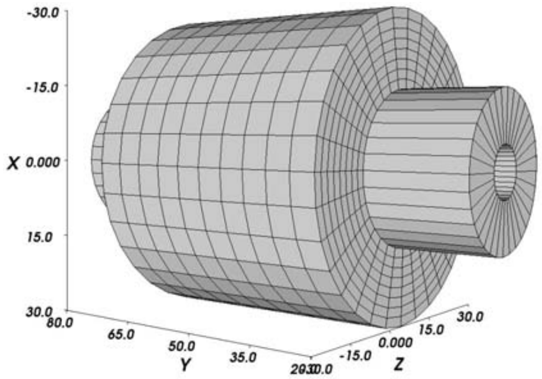


Figure 4: Support. Undeformed configuration.

- Case 2: All potentials of Hencky type: $\mu = 0.19275$ MPa, $\mu^e = 4\mu$, $\eta^v = \tau\mu^e$, $\tau = 0.4$ s. Potential ϕ is null.

Figures 5(b) and 6(b) show the hysteretic loops in force/displacement for the lateral and axial motions respectively. These graphics clearly show the model differences that are accentuated in large strain regimes.

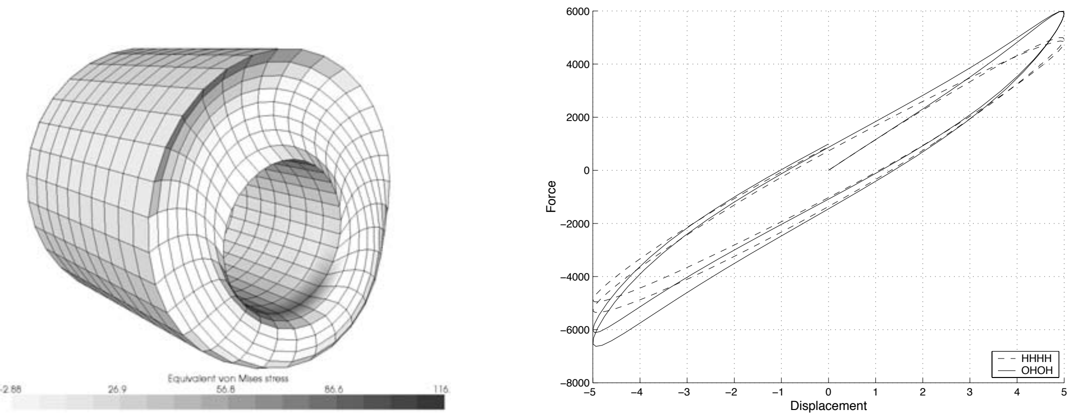


Figure 5: (a) Deformed configuration for lateral edge motion. (b)Lateral force F_x versus displacement u_x.

6 CONCLUSION

We presented in this article a general set of viscoelastic constitutive models based on a variational framework which provides appropriate mathematical structure for further applications like, for example, error estimation. The theoretical and numerical background is stated for general isotropic constitutive functions depending on eigenvalues of strains and strain-rates. As a

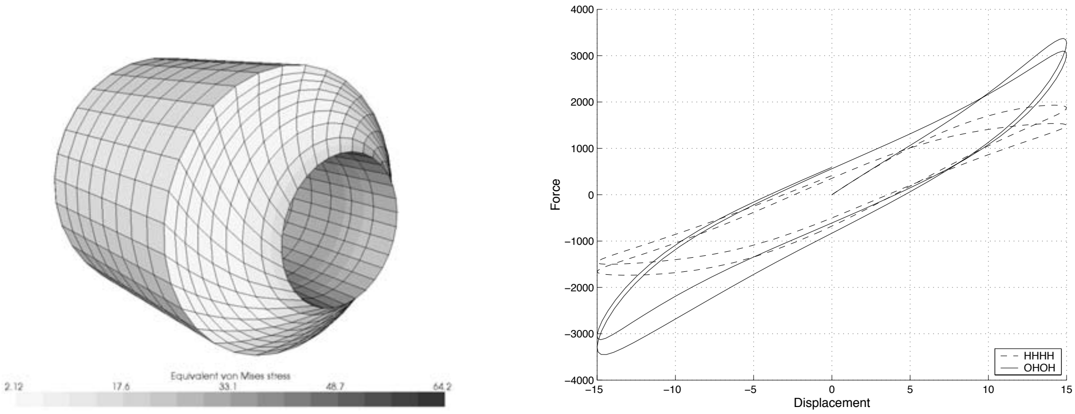


Figure 6: (a) Deformed configuration for axial edge motion. (b) Axial force F_y versus displacement u_y.

consequence, most of the implementation effort, including stress updates and tangent matrix is done at generic level with no relation to a specific isotropic law (potential).

In the numerical examples we compared the capacity of Hencky and Ogden-type models to reproduce observed nonlinear viscous behaviour. As expected, Ogden models perform better for the case of large strains and rubber-like materials, leading to non-elliptic hysteretic loops. Moreover, this choice do not represent, in the present formulation, any additional complexity to the numerical implementation or additional cost on computations.

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