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# POLYNOMIAL CHAOS FOR MODELING MULTIMODAL DYNAMICAL SYSTEMS – INVESTIGATIONS ON A SINGLE DEGREE OF FREEDOM SYSTEM

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**Abstract.** This study aims at pointing out the somehow complex behavior of the structural response of stochastic dynamical systems and consequently the difficulty to represent this behavior using spectral approaches. In the modeling of dynamical systems, uncertainties are present and they must be taken into account to improve the prediction of the models. It is very important to understand how they propagate and how random systems behave. The aim of this work is to find numerically the probability density function (PDF) of response amplitude of random linear mechanical systems when the stiffness is random. Polynomial Chaos performance is first investigated for the propagation of uncertainties in several situations of stiffness variances for a damped single degree of freedom system. For some specific conditions of damping and stiffness variances, it is found that numerical difficulties occur for the Hermite polynomial basis near the resonant frequency. Reasons come from the particular shape of the PDF of the response of the system that can present multimodality. Other bases are then investigated with no better results. Finally a multi-element approach is applied in order to gain robustness.

## 1 INTRODUCTION

This work focuses on the frequency responses of dynamical systems which are subjected to potentially large uncertainties. Common methods for solving stochastic structural dynamics problems are the direct Monte Carlo Simulation (MCS) and the sensitivity-based analysis, such as Neumann or improved perturbations methods. These methods have several drawbacks: MCS is expensive in computing resources for large or complex problems or problems relying on several random variables. Perturbation methods are based on Taylor series expansion of the quantity of interest. The main drawback of these methods is then the small radius of convergence of such series. Hence, efforts are constantly made to explore the suitability of spectral methods, such as the Polynomial Chaos (PC) representation and the Stochastic Reduced Basis Method (SRBM), in order to characterize stochastic mechanical responses. Both of them pertain to a non-statistical approach to represent randomness. Polynomial Chaos representation is based on the "Homogeneous Chaos" theory of Wiener (1938), while SRBM is based on the subspace spanned by the considered application (Nair and Keane, 2002).

In its original formulation, Wiener defined Homogeneous Chaos theory as the span of Hermite polynomial functionals of a Gaussian process. Next, to model uncertainty in physical applications, the continuous integral form of the Hermite-Chaos has been written in the discrete form of infinite summation which is further truncated for computational purpose. This leads to an approximation technique using PC expansion, where square integrable random variables or processes are represented using an Hilbertian basis consisting of Hermite polynomials of independent standard normal variables. The PC expansion provides a complete probabilistic description of the solution. Ghanem and Spanos (1991) combined the Hermite-Chaos expansion with the finite element method to model uncertainties encountered in various problems of mechanics. Xiu and Karniadakis (2002) later extended this strategy by the introduction of the "generalized Polynomial Chaos" (gPC) that includes a broad family of hypergeometric polynomials, the Askey-scheme, to enable a choice for the represention basis. Convergence to any $L_2$ functional in the $L_2$ sense of Askey-scheme based PC expansion is demonstrated in reference Ernst et al. (2012). In contrast, the SRBM chooses a problem-dependent basis. This basis is issued from the vectors which span the stochastic Krylov subspace of the problem. It enables to solve random algebraic systems of equations having non-singular random matrix, such as some of those obtained in structural mechanics. Hence, in both cases (PC expansion and SRBM), exact stochastic solutions are ensured to lie in the (untruncated) subspace generated by the projection basis.

However, the accuracy and effectiveness of the polynomial approximation depend upon the terms that are included in the expansion. Both these approaches will be considered computationally effective when a small number of vectors is used for the expansion. This cannot be achieved for series of slow convergence. Some theoretical results of convergence exist. For any arbitrary random process with finite second-order moments, it is found that the Hermite-Chaos expansion converges with an optimal convergence rate for Gaussian processes, since the weighting function of Hermite polynomials is the same as the Probability Density Function (PDF) of the Gaussian random variables. For other types of processes the convergence rate may be substantially slower with Hermite polynomials; any orthogonal polynomial family, member of the so-called Askey-scheme (Askey and Wilson, 1985), could be a candidate (Xiu and Karniadakis, 2002).

Convergence rate of gPC expansion is studied numerically in references Xiu and Karniadakis (2002); Xiu et al. (2003), where numerical solutions of stochastic ordinary differential equations with different Wiener-Askey chaos expansions are presented. In these works, the choice of the particular Wiener-Askey chaos is based on the distribution of the input random variable. More specifically, references Xiu et al. (2003); Lucor et al. (2004) address the first two moments of second-order ODE associated to a linear oscillator subject to both random parametric and external forcing excitations having three independent random Gaussian variables using Hermite-Chaos. Ten percent is chosen for the coefficient of variation of the input random variables while the nominal system has five percent of damping ratio. It is shown in Xiu et al. (2003) that an expansion order up to 14 is required for a specified error in the standard deviation of the response, while the decay rate for the variance is found lower than for the mean in Xiu et al. (2003); Lucor et al. (2004). Moreover, it is frequent with ODE to observe that the absolute error may increase gradually in time and become unacceptably large for long-term integration. In addition, stochastic regularity of the solution is of the first importance for an efficient approximation using gPC. For discontinuous dependence of the solution on the input random data, gPC may converge slowly or fail to converge even in short-time integration. It is found for linear dynamical systems with closely spaced modes that discontinuities occur in eigenvalues when the random variable evolves (Ghosh and Ghanem, 2008).

These phenomenons may indicate that the chosen basis for the representation of random variables is not appropriate. To maintain a spectral polynomial representation basis, Wan and Karniadakis (2005, 2006) introduce a decomposition of the space of random inputs into small elements where gPC is applied, called multi-element gPC method (MEgPC). In reference Ghosh and Ghanem (2008), another way is adopted, based on an understanding of the physics of the system under consideration, when observing the behavior of a two degrees-of-freedom system having a Gaussian stiffness in one spring, an unphysical hypothesis but appropriate to highlight the main ideas. Some judiciously non-smooth chosen functions, referred as the enrichment functions, are added to the initial Hermite polynomial bases. They are the absolute function, the unit step function and the inverse function.

In Nair and Keane (2002), it is argued that the convergence of the series is intimately related to the overlapping of the probability density functions of the eigenvalues of the stochastic operator for the considered problem. Hence, the practical basis suggested for SRBM in Nair and Keane (2002) comes from a preconditioned stochastic Krylov subspace, performed from the nominal problem. Numerical studies on frequency response analysis of stochastic structural systems is addressed where 40 Gaussian variables are chosen for the members stiffness and mass of a frame structure. In such a situation, the basis is formed by complex vectors. Only the first two moments are studied in this work, showing a degradation in results for large variations of the random variables. However, preconditioning from the nominal problem can not be the optimal choice as it is noticed in Le Maître and Knio (2010), section 7.1.3.2. Two recent developments are noticeable in SRBM. The first one, proposed in Sachdeva et al. (2006), is a hybrid formulation combining SRBM with PC expansions to easily tackle problems that necessitate a large number of basis vectors. The second one is a multi-element formulation of SRBM, proposed in Mohan et al. (2008).

From our experiments in stochastic frequency responses (Pagnacco et al., 2009, 2011), we have found that difficulties can arise with standard PC expansion in some situations, while there is no difficulty for others. Difficulties have been observed for several stiffness distributions (namely the Gaussian, uniform and gamma distributions) in single and multiple degrees-of-freedom systems, having or not other random variables (such as random damping for example). Focusing on the single degree-of-freedom system with only one random variable – the spring stiffness – helps to investigate precisely and understand what is happening and in which con-

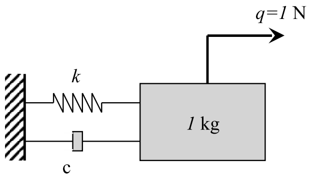


Figure 1: SDoF system

ditions. Analytical and numerical investigations showed that difficulties with PC expansion is intimately related with the potential multimodality that can occur in a specific frequency range, located around the resonant frequency, for some conditions of the variation coefficient of the stiffness when it is compared to the damping ratio. To reduce the size of the paper, we have kept the Gaussian distribution only. Despite the non-physical character of this distribution, this example is easier to understand and helps to focus on main ideas.

Hence, this study focuses on the stochastic frequency response of a Single Degree of Freedom (SDoF) linear oscillator whose stiffness is Gaussian. Some analytical results are first presented in Sec. 2 to understand the behavior of the system response as well as to help for later interpretations of the spectral representation. Sec. 3 provides brief recalls over the PC method and its numerical implementation. Experiments with PC representation come next. First Hermite-Chaos is tested (Sec. 4.1); results are analyzed quantitatively in terms of error over the CDF and qualitatively in terms of CDF and PDF shapes. Then, several other basis are employed (Sec. 4.2), including a MEgPC approach that seems to be the most robust method.

## 2 DESCRIPTION AND ANALYSIS OF THE STOCHASTIC SINGLE-DEGREE-OF-FREEDOM SYSTEM

### 2.1 Description of the deterministic single-degree-of-freedom system

We consider the following SDoF linear oscillator subject to an external harmonic forcing $q(\omega) = 1$ in the frequency domain (Lin, 1967):

$$\left(k - \omega^2 + \mathrm{j}c\omega\right) u(\omega) = 1 \tag{1}$$

where $\omega = 2\pi f$ is the circular frequency associated to the frequency $f$. In this equation, the mass is normalized to unity and $k$ and $c$ are the stiffness and damping parameters of the system. Although being simple, studying such an SDoF system is of interest since it occurs in a similar form when expressing the response of a multiple degrees of freedom system in the modal space. It is sketched in Figure 1.

The system response $u(\omega)$ is given by:

$$u(\omega) = g(k;\, \omega) = \frac{1}{k - \omega^2 + \mathrm{j}c\omega} \tag{2}$$

which is a complex quantity such that:

$$\mathrm{Real}\left[u(\omega)\right] = \left(k - \omega^2\right) |u(\omega)|^2 \qquad \text{and} \qquad \mathrm{Imag}\left[u(\omega)\right] = -\omega c\, |u(\omega)|^2 \tag{3}$$

where $\mathrm{Real}[\bullet]$ and $\mathrm{Imag}[\bullet]$ denote the real and imaginary part, respectively, while $|u(\omega)|$ is the response amplitude given by:

$$|u(\omega)| = \frac{1}{\sqrt{\left(k - \omega^2\right)^2 + (\omega c)^2}} \tag{4}$$

The response amplitude $|u(\omega)|$ has a maximum at the resonant frequency which is close to the natural frequency of the SDoF system given by:

$$f_{\mathrm{n}} = \frac{1}{2\pi}\sqrt{k} \tag{5}$$

## 2.2 Description of the stochastic single-degree-of-freedom system

Let us consider now a probability space $(\Omega,\, \mathcal{A},\, \mathrm{Prob})$ with $\Omega$ the event space, $\mathcal{A}$ the $\sigma$-algebra on $\Omega$, and $\mathrm{Prob}$ a probability measure. The SDoF system becomes stochastic if at least one of its parameters is random. Let us consider the situation when only the stiffness constant $k$ is random. Random variables will be denoted by the capital letter which matches the deterministic variable, hence $K$ in this case. It is such that $K(\varpi) : \Omega \longrightarrow \mathbb{R}$ where $\varpi \in \Omega$. Let $P_K(k) = \mathrm{Prob}\,[K \leq k]$ be the Cumulative Distribution Function (CDF) and $p_K(k) = \frac{\mathrm{d}P_K}{\mathrm{d}k}$ denote the PDF of $K$ having $\mu_K$ as the mean and $\sigma_K$ as the standard deviation. The domain of $K$ is considered to be an interval having the boundaries $k_{\mathrm{inf}}$ and $k_{\mathrm{sup}}$ that may or may not belong to the interval depending on the chosen distribution for $K$. For the Gaussian PDF, $k \in \,]k_{\mathrm{inf}},\, k_{\mathrm{sup}}[$ with $k_{\mathrm{inf}} = -\infty$, $k_{\mathrm{sup}} = +\infty$ , and the interval is open, the boundaries do not belong to the interval. For such a system, the natural frequency of the SDoF system is also a random variable, given by $F_n = \frac{1}{2\pi}\sqrt{K}$.

Interest arises now in the random system response $U(\varpi;\, \omega)$ that we can also denote $U(\omega)$ in the following, knowing the dependency in $\varpi$ by the use of the capital letter. It is a complex process which depends on the circular frequency $\omega$, given by:

$$\left(K - \omega^2 + \mathrm{j}c\omega\right) U(\omega) = 1 \tag{6}$$

that we can compactly rewrite as:

$$U(\omega) = g(K;\, \omega) \tag{7}$$

for:

$$g(K;\, \omega) = \frac{1}{K - \omega^2 + \mathrm{j}c\omega} \tag{8}$$

where $g$ is a non-linear function of its variables. Then, the system response is such that:

$$U(\omega) = \mathrm{Real}\,[U(\omega)] + \mathrm{j}\mathrm{Imag}\,[U(\omega)] \tag{9}$$

where the real and imaginary parts are random processes:

$$\mathrm{Real}\,[U(\omega)] = \left(K - \omega^2\right)|U(\omega)|^2 \qquad \text{and} \qquad \mathrm{Imag}\,[U(\omega)] = -\omega c\,|U(\omega)|^2 \tag{10}$$

and the amplitude is:

$$|U(\omega)| = \frac{1}{\sqrt{\left(K - \omega^2\right)^2 + (\omega c)^2}} \tag{11}$$

Its moments are defined from the expectation:

$$\mathrm{E}\,[h(U)] = \int_{u_{\mathrm{inf}}}^{u_{\mathrm{sup}}} h(u)\,\mathrm{d}P_U(u,\, \omega) = \int_{u_{\mathrm{inf}}}^{u_{\mathrm{sup}}} h(u)\,p_U(u,\, \omega)\,\mathrm{d}u \tag{12}$$

where $p_U(u,\, \omega)$ is the complex marginal PDF of the process $U(\omega)$. Then, we define the mean as $\mu_U(\omega) = \mathrm{E}\,[U(\omega)]$ and the variance $\sigma^2_{UU^*}(\omega) = \mathrm{E}\,[(U(\omega) - \mu_U(\omega))\,(U^*(\omega) - \mu^*_U(\omega))]$ and the relation $\sigma^2_{UU}(\omega) = \mathrm{E}\left[(U(\omega) - \mu_U(\omega))^2\right]$ where $\bullet^*$ denotes the complex conjugate.

### 2.3 Analysis of the stochastic single degree of freedom system

In this section, emphasis is given to the complexity of the shape of the PDFs of the SDoF system response. Our experience suggests that the complexity is intimately related to the number of statistical modes of the distribution and its potential asymmetry. As a consequence, the focus is on the existence of several statistical modes which finally depends on the system parameters and the frequency range of interest.

To set these ideas, let the stiffness K be a random variable having Gaussian distribution with mean $\mu_K$ and variance $\sigma_K^2$:

$$p_K(k) = \frac{1}{\sigma_K\sqrt{2\pi}} \exp\left(-\frac{(k-\mu_K)^2}{2\sigma_K^2}\right) \tag{13}$$

In this case $k_{\text{inf}} = -\infty$ and $k_{\text{sup}} = +\infty$, but it is obvious that the probability of $k$ being negative can be negligible for an adequate choice of the mean and the variance.

To express the system response analytically, it is necessary to distinguish the static case from the dynamic one. For the static case, $\omega = 0$ and $U$ is real. There is one single root $k_1$ for the algebraic equation $u(k,\omega) = u$, being $k_1(u) = \frac{1}{u}$. Then, the system response PDF at a fixed frequency being the marginal PDF of the process, it is found (Zwillinger and Kokoska, 2000) to be:

$$p_U(u, \omega = 0) = \frac{1}{u^2}\frac{1}{\sigma_K\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{1}{u}-\mu_K\right)^2}{2\sigma_K^2}\right) \tag{14}$$

From this analytical expression of the PDF, when limiting the analysis to the meaningful positive part[1], it can be observed that it always leads to a bell shape having a positive skewness (the right tail is longer) and a positive excess kurtosis[2]. Then, the shape of the static response is simple, with only one statistical mode (located at $u_1 = \frac{1}{4}\left(\sqrt{\mu_K^2 + 8\sigma_K^2} - \mu_K\right)$).

For the dynamic case, $\omega > 0$ is considered and it is such that $k_{\text{inf}} < \omega^2 < k_{\text{sup}}$. Thus, there are two roots $k_1$ and $k_2$ for the algebraic equation $|u(k,\omega)| = |u|$ when $\omega$ is fixed, being $k_{1,2}(u) = \mp\frac{1}{u}\sqrt{1 - 4\eta^2\mu_K\omega^2u^2} + \omega^2$ when damping ratio $\eta = \frac{c}{2\sqrt{\mu_K}}$ is introduced. In this case, the system amplitude response PDF is given by Zwillinger and Kokoska (2000):

$$p_{|U|}(u,\omega) = \frac{p_K(k_1) + p_K(k_2)}{u^2\sqrt{1-4\eta^2\mu_K\omega^2u^2}} \tag{15}$$

To find the support of this PDF, we consider the relation (11). Since $K \in \left]-\infty, +\infty\right[$, $\left(K-\omega^2\right)^2 \in \left]0, +\infty\right[$, $\sqrt{\left(K-\omega^2\right)^2 + \left(2\sqrt{\mu_K}\eta\omega\right)^2} \in \left]2\eta\sqrt{\mu_K}\omega, +\infty\right[$, and $|U(\omega)| \in \left]0, \frac{1}{2\eta\sqrt{\mu_K}\omega}\right[$, exhibiting finite bounds for a damped system. Hence, the PDF of the imaginary part of the response has also finite bounds, at the contrary of the PDF of the real part.

For low frequencies $\omega$, the PDFs of the amplitude and the real part of the response are nearly similar to the one found in the static case, as it is expected. For high frequencies, all the PDFs

[1] Let us recall the working conditions: the forcing is chosen positive while the stiffness properties are chosen such that the probability to be negative is negligible.

[2] Notice that its moments do not exist theoretically, but approximations that agree with empirical results can be found: $\mu_U^{-1} \simeq \mu_K\left(1 - \frac{\sigma_K^2}{\mu_K^2}\right)$ for the mean and $\frac{c_K}{\sqrt{\mu_K^2+\sigma_K^2}} \leq \sigma_U \leq \frac{c_K^2}{\mu_K\left(1+c_K^2\right)^2\left(1-2c_K^2\right)}$ for the standard deviation of the response, having defined $c_K = \frac{\sigma_K}{\mu_K}$.

(for real, imaginary and amplitude of the response) have a bell shape like a Gaussian distribution with a quasi-null skewness and a quasi-null excess kurtosis.

For the fixed frequency $\omega = \sqrt{\mu_K}$ which corresponds to the natural frequency of the nominal system, the above expression simplifies:

$$p_{|U|}\left(u,\omega\right) = \frac{1}{u^2\sqrt{1-4\eta^2\mu_K^2u^2}} \times \frac{2}{\sigma_K\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{1}{u}\sqrt{1-4\eta^2\mu_K\omega^2u^2}\right)^2}{2\sigma_K^2}\right) \tag{16}$$

enabling more insight into the result. This reveals the existence of two potential statistical modes for the amplitude response PDF. The first one can occur when the following condition[3]:

$$c_\eta = \frac{\sigma_k}{\eta\mu_K} > \frac{2}{1+\sqrt{3}} \tag{17}$$

is fulfilled for $c_\eta = \frac{\sigma_K}{\eta\mu_K}$ that we call here the "normalized coefficient of variation". The second mode, located at the supremum bound of the distribution, occurs systematically. Notice also that the PDF tends to have an infinite slope at this bound.

The analysis given for the shape of the PDF of the response amplitude at the specific frequency $\omega = \sqrt{\mu_K}$ is sufficient to conclude for the shape of the PDFs of the real and the imaginary part of the response. Indeed, relations (3) reveal that the real part corresponds to the multiplication of the square of the amplitude with the law chosen for the stiffness when it is translated to be centered, i.e. a Gaussian centered distribution. Then, considering the shape in positive values of the PDF of the real part, one can argue that it is a distortion of the PDF of the amplitude that keeps unchanged the number of statistical modes. Moreover, as a consequence of the symmetry of the centered Gaussian, the shape found for the positive values is mirrored in negative values, doubling thus the number of statistical modes. Ideas are more simple for the imaginary part since it is only an amplified square of the amplitude which thus leads to a distorted but similar shape, having the same number of modes. To illustrate this, responses of a SDoF system are considered for several damping ratios $\eta$ and keeping constant uncertain parameters $\mu_K$ and $\sigma_K$. It follows that each choosen value for the damping ratio corresponds to a different normalized coefficient of variation $c_\eta$ and hence, to a different complex stochastic response. Figures 2 and 3 show the evolution in shape for the PDFs of the real and the imaginary parts when the normalized coefficient of variation $c_\eta$ of the SDoF system is progressively increased (that is when the damping ratio is progressively decreased).

Shape analysis is less intuitive for frequencies around $\omega = \sqrt{\mu_K}$. In this case, the shape of the PDFs of the amplitude and of the imaginary part of the response evolve slowly, but the symmetry in the shape of the PDF of the real part is clearly lost due to the presence of the term $K - \omega^2$ in relation (3). This potentially leads to less statistical modes when the condition (17) is fulfilled, but there is an increasing complexity of the shape.

Finally, the shape of the response over all frequencies for an SDoF system can be quite simple or not, depending on the system parameters values. Concerned parameters are the stiffness mean and variance and the damping level. For a moderately damped system having a non negligible uncertainty, the PDFs of the response can evolve significantly from a simple shape in low frequencies to a complex one near the resonant frequency, while it becomes very simple when reaching high frequencies. Hence, it can be asserted that a Polynomial Chaos representation of

[3]Note that in the undamped case, the PDF has an open right bound, being the infinite, where this statistical mode is located in any case.

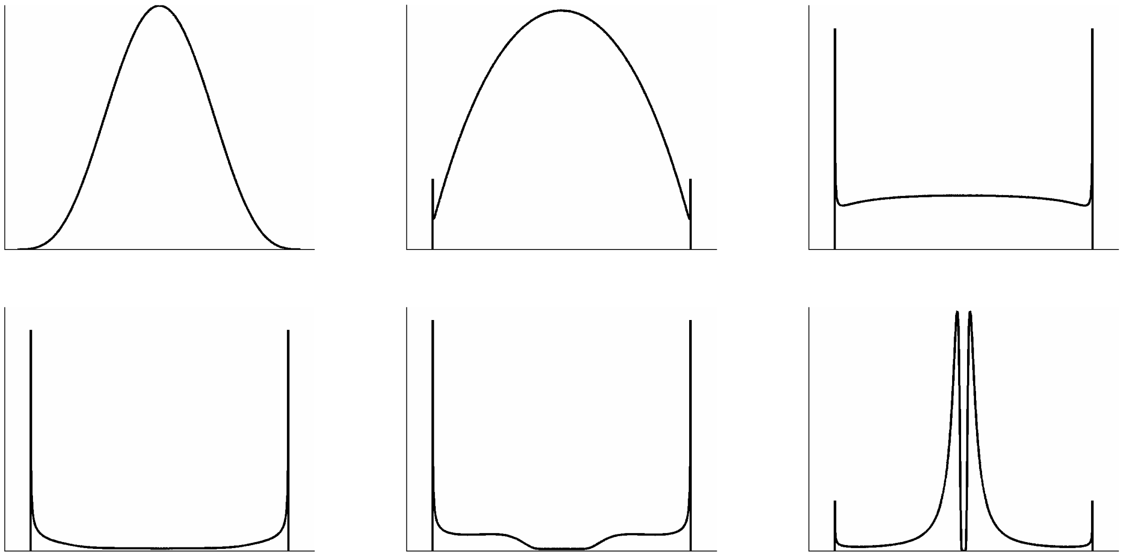

Figure 2: PDFs of the real part of the responses of 6 SDoF systems having different normalized coefficient of variation $c_\eta$: from the left most hand, up, to the right most hand, down, they are: 0.33, 0.57, 0.76, 2.3, 5.7 and 57. Stiffness coefficient of variation is fixed, and the natural frequency of the nominal system is considered.

this response may be inefficient near the resonant frequency. In addition, coming back to the results reported in the literature (Nair and Keane, 2000; Xiu et al., 2003; Lucor et al., 2004), this can explain the bad estimation of the second moment observed in this range of frequencies for a system having these parameters.

In the following, since the stochastic process of interest does not belong to the class of differential stochastic processes, we can focus directly on fixed values for $\omega$. Then, we will not consider the random process $U(\varpi; \omega)$ but several distinct random variables $U_\omega(\varpi)$.

## 3 PROPAGATION OF UNCERTAINTY USING A POLYNOMIAL CHAOS REPRESENTATION

Only the principle is recalled here for a dimension-one stochastic space, that is when only one random variable $\xi$ is used to introduce randomness in the system. The reader is referred to the references cited in the introduction for a complete presentation of PC method.

Considering a second-order random process $X$, the PC expansion proposes to express it as a polynomial series using a set of $n_x$ orthogonal polynomials denoted $\psi_p$ in the variable $\xi$:

$$X(\varpi) = \sum_{p=0}^{n_x-1} x_p \psi_p(\xi(\varpi)) \tag{18}$$

the order $n_x$ being theoretically infinite for general situations. The deterministic coefficients $x_p$ now used to represent $X$ can be evaluated in two ways: using an intrusive method or a non-intrusive one. The intrusive method follows a Galerkin approach: expression (18) is introduced in the equations governing $X$ and theses equations are projected onto the set of orthogonal polynomials $\psi_p$. The non-intrusive method uses the orthogonality of the polynomials with

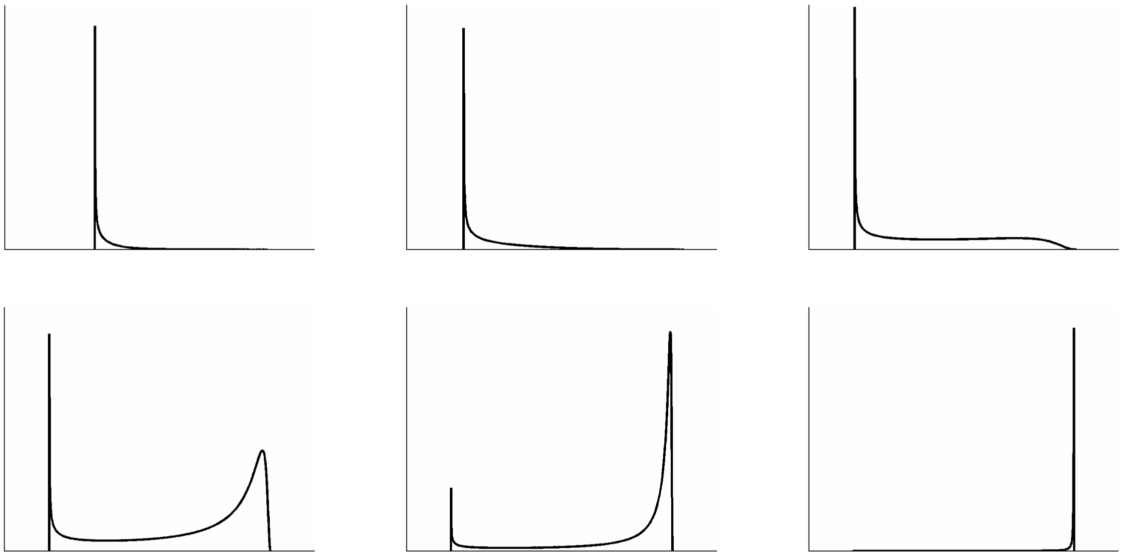

Figure 3: PDFs of the imaginary part of the responses of 6 SDoF systems having different normalized coefficient of variation $c_\eta$: from the left most hand, up, to the right most hand, down, they are: 0.33 0.76 2.3 5.7 11 and 57. Stiffness coefficient of variation is fixed, and the natural frequency of the nominal system is considered.

respect to the scalar product $< \bullet, \bullet >$ to evaluate the coefficients $x_p$:

$$x_p = \frac{< X, \psi_p >}{< \psi_p, \psi_p >} \tag{19}$$

where the numerator at least is evaluated using a quadrature rule. The main difference between both methods is that the intrusive methods provides a set of $m \times n_x$ coupled algebraic equations (where $m$ is the size of the underlying deterministic problem) and often requires a special implementation while the non-intrusive approach determines the set of coefficients $x_p$ one after the other in an independent manner and reuses existing codes to evaluate $X$ values needed for the quadrature.

## 4 APPLICATION OF POLYNOMIAL CHAOS AND OTHER METHODS TO SDOF SYSTEM

The chosen distribution for the input variable $K$ is a Gaussian or normal law with mean $\mu_K$ and standard deviation $\sigma_K$:

$$p_K(k) = \frac{1}{\sigma_K\sqrt{2\pi}} e^{-\frac{(k-\mu_K)^2}{2\sigma_K^2}} \tag{20}$$

According to the results of the gPC theory, it is known that Hermite polynomials $\psi_p$ are the optimal choice for the expansion basis of the input variable. Indeed, it leads to the exact two terms expansion given by:

$$K = \sum_{p=0}^{n_K-1} k_p \psi_p(\xi) = \mathbf{\Psi}_H(\xi)\,\mathbf{k} \tag{21}$$

with

$$\{\mathbf{k}\}_p = \frac{\left\langle \{\mathbf{\Psi}_H\}_p K \right\rangle}{\left\langle \{\mathbf{\Psi}_H\}_p \{\mathbf{\Psi}_H\}_p \right\rangle} \tag{22}$$

where $\langle \bullet \rangle = \int_{\mathbb{R}} \bullet(\xi)\, p_\xi(\xi)\, \mathrm{d}\xi$ for $\xi$ which follows a normal law having a zero mean and a unit standard deviation. This leads to $n_K = 2$ such that $k_0 = \mu_K$ and $k_1 = \sigma_K$.

### 4.1 Hermite polynomial representation of the response

In a standard intrusive approach, when following the strategy proposed in Ghanem and Spanos (1991); Xiu and Karniadakis (2002, ...), the output variable is expanded onto the same basis than the input variable. In the present case, it is a Hermite basis, and the output variable is the system response $U_\omega$ for a given circular frequency $\omega$. This leads to represent $U_\omega$ as:

$$U_\omega = \sum_{q=0}^{n_u-1} u_q \psi_q^H(\xi) = \mathbf{\Psi}_H(\xi)\, \mathbf{u} \tag{23}$$

with the complex coefficients $u_q$ for a basis $\mathbf{\Psi}_H$ composed of Hermite polynomials up to degree $n_u - 1$. It has to be noticed that such a limited expansion where higher orders are truncated leads generally to an approximation. Another important remark concerns the interpretation of the coefficients $u_q$: while the mean of the real part and of the mean of the imaginary part of $U_\omega$ are both given by the complex coefficient $u_0$, the mean of $|U_\omega|$ as well as higher order moments of $U_\omega$ or $|U_\omega|$ are expressed as a combination of all the coefficients $u_q$. Hence, a non appropriate order of truncation or a misevaluation of the expansion coefficients leads implacably to a drift in theses moments. As a consequence, fidelity in the PC representation of the PDFs of the real and imaginary parts of $U_\omega$ is necessary to correctly estimate the mean of the amplitude.

By following the intrusive approach, the combination of the two above expansions for the input and the output random variables with the dynamic equation for the SDoF system leads to define the error $e(\xi)$ as:

$$e(\xi) = \left(\mathbf{\Psi}_H(\xi)\, \mathbf{k} - \omega^2 + \mathrm{j}c\omega\right) \mathbf{\Psi}_H(\xi)\, \mathbf{u} - 1 \tag{24}$$

where unknown coefficients vector $\mathbf{u}$ can be obtained by projecting the error $e(\xi)$ onto the trial basis:

$$\langle e(\xi), \psi_q(\xi) \rangle = \int_{\mathbb{R}} e(\xi)\, \psi_q(\xi) p_\xi\, \mathrm{d}\xi = 0, \qquad q = 1,\, 2,\, \cdots,\, n_u \tag{25}$$

where $\langle e(\xi), \psi_q(\xi) \rangle$ is the inner product in the Hilbert space determined by the support of the random variable $\xi$. This leads to the following set of $n_u$ equations for the estimation of $\mathbf{u}$:

$$\left[ \sum_{p=0}^{n_K-1} \{\mathbf{k}\}_p \left\langle \mathbf{\Psi}_H^{\mathrm{H}} \{\mathbf{\Psi}_H\}_p \mathbf{\Psi}_H \right\rangle + \left(-\omega^2 + \mathrm{j}c\omega\right) \left\langle \mathbf{\Psi}_H^{\mathrm{H}} \mathbf{\Psi}_H \right\rangle \right] \mathbf{u} = \left\langle \mathbf{\Psi}_H^{\mathrm{H}} \right\rangle \tag{26}$$

or:

$$\left[\sigma_K \left\langle \mathbf{\Psi}_H^{\mathrm{T}} \xi \mathbf{\Psi}_H \right\rangle + \left(\mu_K - \omega^2 + \mathrm{j}c\omega\right) \left\langle \mathbf{\Psi}_H^{\mathrm{T}} \mathbf{\Psi}_H \right\rangle\right] \mathbf{u} = \left\langle \mathbf{\Psi}_H^{\mathrm{T}} \right\rangle \tag{27}$$

where $\bullet^{\mathrm{H}}$ denotes the hermitian, that simplifies to the transpose when using real polynomials, as it is the case here.

As mentioned previously, the accuracy of the expansion strongly depends on the ratio between the stiffness dispersion (measured by $\sigma_K/\mu_K$ ratio) and the damping. Hence, several

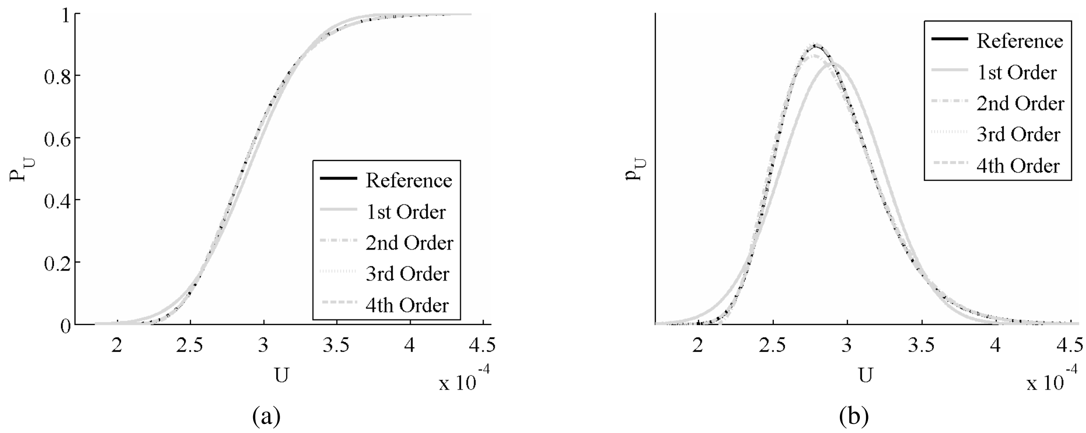


Figure 4: Comparison of empirical CDF (a) and PDF (b) obtained for $U_0$ with different expansion orders onto an Hermite basis at the null frequency (*i.e.* the static case)

configurations are tested numerically. To this end, considered numerical values are arbitrarily chosen such that: $\mu_K = 3500$ N/m, $\sigma_K = 400$ N/m and $\eta_m \in \{0.35,\ 0.05,\ 0.02\}$. This leads to a normalized coefficient of variation $c_\eta = \frac{\sigma_K}{\eta_m \mu_K} \simeq \{0.33,\ 2.3,\ 5.7\}$.

The first numerical investigation with PC expansion concerns the static case. This situation is the most common of the literature. Not surprisingly, Figure 4 shows that extremely satisfactory results can be achieved at the 4th order. Regarding the convergence of the series, it is observed that the CDFs and PDFs approach better and better the reference (that is indifferently given by a MCS using a $10^6$ sample or the formula (14)) curve from an order of expansion to the next one. This behavior for the CDF approximation is in accordance with theoretical results of PC expansion. But for the PDF, from the results reported in the following of this study, we state that this behavior is only a consequence of the low order $n_u$ required to get an accurate result.

The mean quadratic norm of the empirical CDF error $\epsilon^2_{\text{CDF}}$ helps to quantify the expansion quality for each frequency of the range of interest. Since the random variable of interest is a complex quantity, that is potentially unbounded, we propose to use the amplitude of the system response in order to define this error as:

$$\epsilon^2_{\text{CDF}}\left(U_\omega\right) = \frac{1}{\mathrm{E}\left[U_\omega\right]^2} \int_{\inf[U_\omega]}^{\sup[U_\omega]} \left(\widehat{P}_{|U_\omega|}\left(u\right) - P_{|U_\omega|}\left(u\right)\right)^2 \mathrm{d}u \tag{28}$$

where $\widehat{P}_{|U_\omega|}\left(u\right)$ denotes the empirical CDF of the distribution obtained from the polynomial expansion, while $P_{|U_\omega|}\left(u\right)$ is the reference CDF. From this definition, it is supposed that a satisfactory representation is achieved when $\epsilon^2_{\text{CDF}}\left(U_\omega\right)$ is less than or equal to $0.3\ \%$ (that is $\epsilon_{\text{CDF}}\left(U_\omega\right) \leq 5.5\ \%$ ), meaning that the true PDFs of the response agreewell with their PC expansion. However, due to numerical stability reasons, we limit the order of the expansion to the 60th order.

Figure 5 shows the order of truncation required to achieve a satisfactory representation of the PDFs using the above criterion. Results for the three cases of SDoFs systems studied are represented over the normalized frequencies. Normalized frequency is defined by the ratio between the true frequency and the resonant frequency of the nominal SDoF system. As it is expected, very satisfactory results are again obtained at high frequency (saying a normalized frequency equal to two in the current example) with an expansion truncated at the 3rd order,

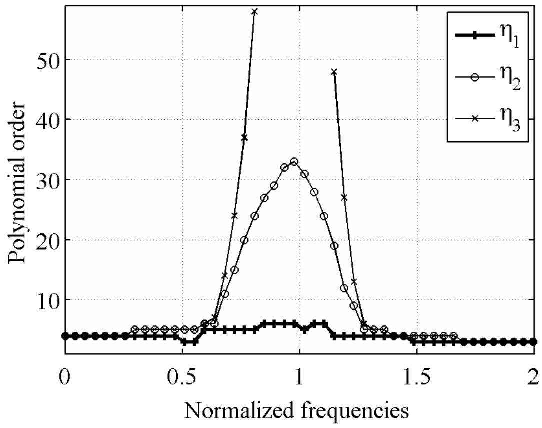


Figure 5: Order of polynomial required versus normalized frequencies to satisfy $\epsilon^2_{\text{CDF}} \leq 0.3\%$ for the Hermite polynomial representation of the SDoF response when considering the three damping ratios $\eta_m$, $m \in \{1,\ 2,\ 3\}$.

whatever the damping level is chosen. Looking at the PDFs of the real and the imaginary parts of $U_\omega$ on logarithmic scale shows that the two last coefficients of the Hermite representation helps to well adjust the tails of the distributions. For the high damping case $\eta_1$, it is found that a 6th order expansion suffices for a good representation over the complete frequency range and especially around the resonant frequency of the nominal system. For the medium damping case $\eta_2$, the Hermite polynomial representation is not really satisfactory since a 33rd order is required to achieve a sufficient quality around the resonant frequency of the nominal system.

For the low damping $\eta_3$, results are disastrous since the maximal order for the PC expansion – 60th order in this study – does not lead to accurate results with $\epsilon_{\text{CDF}}\left(U_\omega\right) \simeq 22\ \%$. This produces a region with no satisfactory results around the resonant frequencies on the graph. It can be explained by a very poor convergence of the Hermite expansion for the SDoF response at these frequencies for such a normalized coefficient of variation $c_\eta$. To gain insight into such a failure of the PC expansion, the Figure 6 shows the empirical CDFs and PDFs of real and imaginary parts obtained for the PC expansion of $U_\omega$ at the 60th order, considering the unitary normalized frequency. It is observed that both CDFs seems almost satisfactory, so that they can lead to converged values for the firsts statistical moments. However, a major discrepancy occurs for the CDF of the real part if we look at its extrema, indicating that probabilities of low and high levels are certainly incorrect for this PC expansion. Another major discrepancy occurs for the distribution of the imaginary part since it has a tail fragment into the positive values region of the displacements. In addition, there are many oscillations which are easily revealed by the observation of both PDFs. This observation goes at the contrary of the one made for the static case where it has been observed that the PDFs of a non converged expansion is anyhow closed to the reference curve. This is a consequence of using very high orders: there are many spurious peaks, and they do not reflect the real ones, since they do not occur systematically at the correct location.

## 4.2 Representations using other bases

### 4.2.1 Using standard bases

To cope with a satisfactory expansion of the response when a high normalized coefficient of variation $c_\eta$ characterizes the system of interest, the simplest idea is to try several polynomial

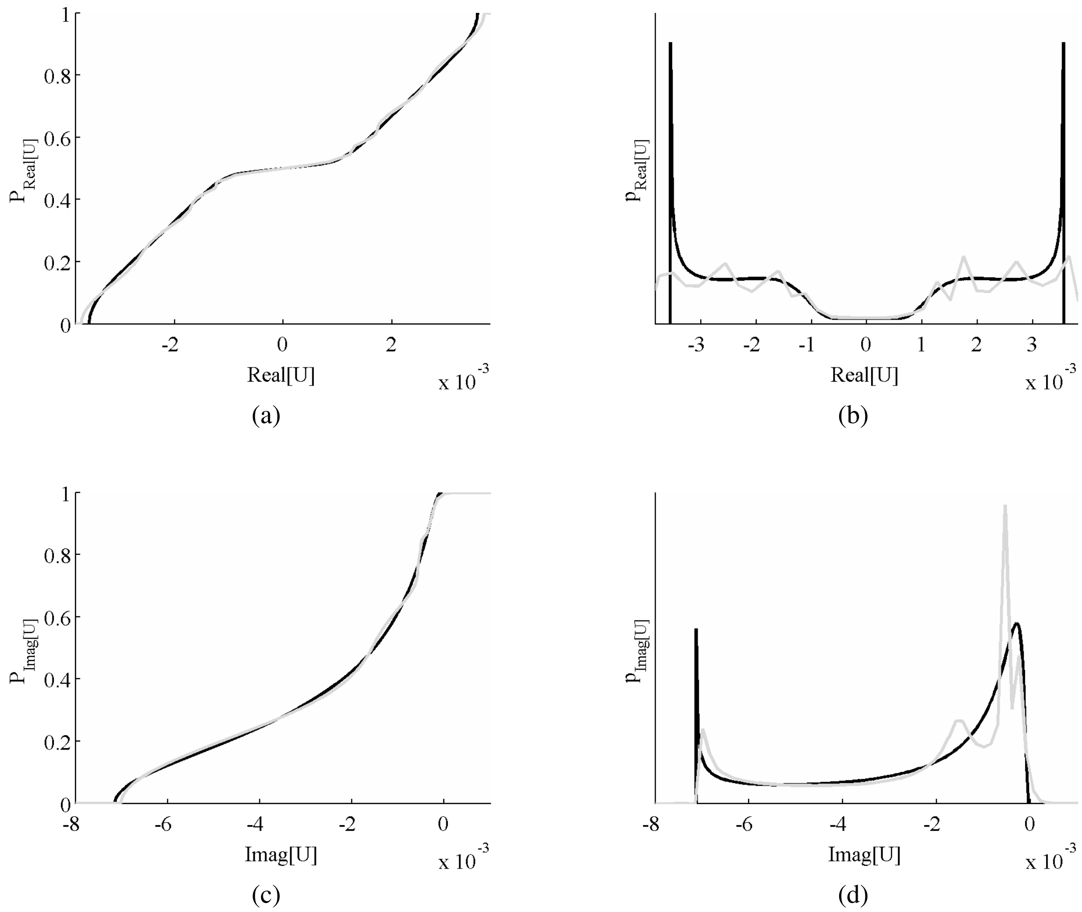


Figure 6: Comparison of empirical CDFs of (a) real and (b) imaginary parts and PDFs of (c) real and (d) imaginary parts obtained for $U_\omega$ with a 60th order expansion onto an Hermite basis at the natural frequency of the nominal system when $\eta_3$; black curves are the reference, grey curves are the PC representation.

bases. Since it is a continuous distribution, Legendre, Laguerre and Chebyshev polynomials are first candidates.

The non intrusive approach then necessitates the use of an isoprobabilistic transformation (Xiu and Karniadakis, 2002) to link the Gaussian random variable $\xi$ and a random variable $\zeta$ following another law, appropriated to the chosen polynomial basis. Let us focus on the Legendre basis, whose polynomials are denoted $\psi_q^L$. According to the literature, $\zeta$ should follow a uniform law over $[-1, 1]$. Let us denote $\mathcal{T}$ the function that performs the isoprobabilistic transformation from $\zeta$ to $\xi = \mathcal{T}(\zeta)$. For such a situation:

$$U_\omega = \sum_{q=0}^{n_u-1} u_q \psi_q^L(\zeta) = \mathbf{\Psi}_L(\zeta)\,\mathbf{u} \tag{29}$$

with

$$u_q = \frac{\langle \psi_q^L U_\omega \rangle}{\langle \psi_q^L \psi_q^L \rangle} \tag{30}$$

where $\langle \bullet \rangle = \frac{1}{2}\int_{-1}^{1} \bullet(\zeta)\, d\zeta$. As stated in Sec. 3, the numerator is evaluated through an appropriated quadrature rule, that is a Gauss-Legendre quadrature in the developed case :

$$\langle \psi_q^L U_\omega \rangle = \sum_{i=1}^{n_{GL}} \psi_q^L(\zeta_i) U_\omega(\zeta_i)) w_i \tag{31}$$

being (from relation (8)):

$$\langle \psi_q^L U_\omega \rangle = \sum_{i=1}^{n_{GL}} \psi_q^L(\zeta_i) g\left(K\left(\mathcal{T}\left(\zeta_i\right)\right); \omega)\right) w_i \tag{32}$$

with $n_{GL}$ the number of $\zeta_i$ evaluation points for the quadrature and $w_i$ their weights. The isoprobabilistic transformation is then necessary to evaluate $K$ for the given $\zeta_i$.

Following this way, and focusing only on the unitary reduced frequency in a non intrusive approach, it is found experimentally when considering $\eta_3$ for the investigated application that the 13th order for a Legendre basis is the minimal order which leads to an accurate enough representation (according to the $\epsilon_{\text{CDF}}$ criterion). Figure 7 summarizes results of necessary orders found over the full range of frequencies with this basis.

That said, an intrusive approach is desirable for linear problems. Following the above intrusive method (Sec. 4) and replacing the Hermite basis $\mathbf{\Psi}_H$ by the Legendre basis $\mathbf{\Psi}_L$ however does not lead to any better results, if we keep the order $n_K$ lower or equal to $n_U$; on the contrary, results are worse as higher orders are required for a same accuracy, as it is seen in Figure 8. Explanation comes from the fact that in the classical intrusive formulation of the PC approach, the polynomial basis used to expand the input random variable is the same as the one used to expand the output random variable. This means that $K$, which is a Gaussian random variable should first be expanded over the Legendre basis using the previously mentionned isoprobabilistic transformation $T$:

$$K = \sum_{p=0}^{n_K-1} k_p \psi_p^L(\zeta) = \mathbf{\Psi}_L(\zeta)\,\mathbf{k}, \quad k_p = \frac{\langle \psi_p^L K \rangle}{\langle \psi_p^L \psi_p^L \rangle}, \quad \langle \psi_p^L K \rangle = \sum_{i=1}^{n_{GL}} \psi_p^L(\zeta_i) K(\mathcal{T}(\zeta_i)) w_i \tag{33}$$

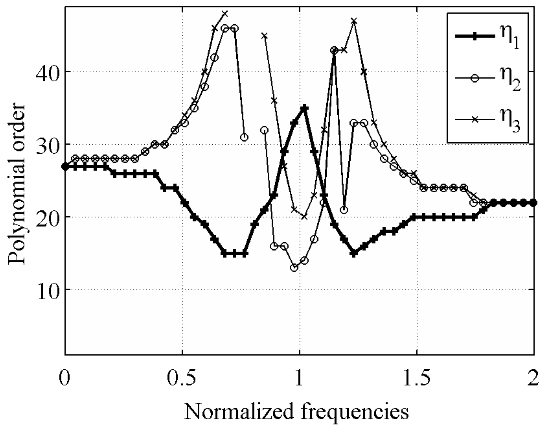


Figure 7: Order of polynomial required to satisfy $\epsilon^2_{\mathrm{CDF}} \leq 0.3\%$ versus normalized frequencies for the Legendre polynomial representation of the SDoF response when considering the three damping ratios $\eta_m$, $m \in \{1,\, 2,\, 3\}$.

Yet, the Legendre basis truncated at the order 13th does not lead to a good expansion of the current stiffness, as it is seen in Figure 9a) and Figure 9b): even if the CDF of $K$ seems good enough at this expansion order, the PDF shows that convergence is not truly achieved. Moreover, the found ripple phenomena will persist for higher orders, as it is shown for the 34th order in Figure 9c). Then, a remedy would be to increase the number of terms for the representation of $K$, being more higher than the number necessary for $U_\omega$.

#### 4.2.2 Formulation and results of a two bases PC representation

An alternative for an intrusive approach with the Legendre basis can be found from a non classical way, by using two different expansion bases in the same formulation. Indeed, we decide to expand the input variable $K$ onto an Hermite basis $\mathbf{\Psi}_H$ while the output variable $U_\omega$ is expanded onto the Legendre basis $\mathbf{\Psi}_L$, through an appropriate formulation. This enables to keep the two expansion orders, the one for the input variable and the one for the output variable, as low as possible.

The first step of this approach consists in finding the expansion coefficients of $K$ from a projection, by using here the Hermite-Gauss quadrature integration, as in Sec. 4:

$$K = \mathbf{\Psi}_H\left(\xi\right)\mathbf{k} = \sum_{p=0}^{n_K-1} \left\{\mathbf{\Psi}_H\right\}_p \left\{\mathbf{k}\right\}_p \tag{34}$$

Then, both expansions of the input variable $K$ and output variable $U_\omega$ are introduced into the dynamic relation, as usual, but each using its own basis. To recover expansion coefficients of the output variable, the resulting system is projected onto the basis chosen for the output variable, with its numerical quadrature scheme to express the expectations, being the Legendre-Gauss quadrature integration for the current application:

$$\left[\sum_{p=0}^{n_K-1} \left\{\mathbf{k}\right\}_p \left\langle \mathbf{\Psi}_L^{\mathrm{T}} \left\{\mathbf{\Psi}_H\right\}_p \mathbf{\Psi}_L \right\rangle + \left(-\omega^2 + \mathrm{j}c\omega\right)\left\langle \mathbf{\Psi}_L^{\mathrm{T}} \mathbf{\Psi}_L \right\rangle\right] \mathbf{u} = \left\langle \mathbf{\Psi}_L^{\mathrm{T}} \right\rangle \tag{35}$$

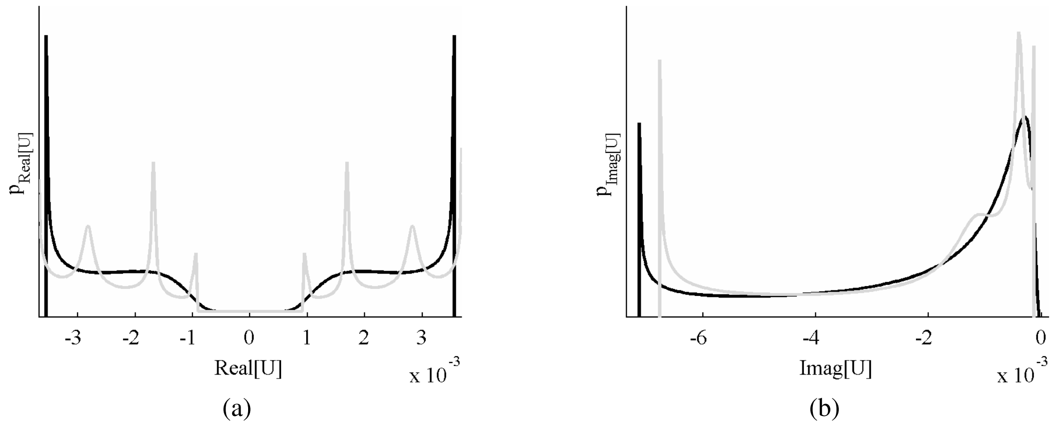


Figure 8: Comparison of empirical PDFs of (a) real and (b) imaginary parts obtained for $U_\omega$ with a 13th order expansion onto a Legendre basis at the natural frequency of the nominal system for $\eta_3$ using a classical intrusive formulation; black curves are the reference, grey curves are the PC representation.

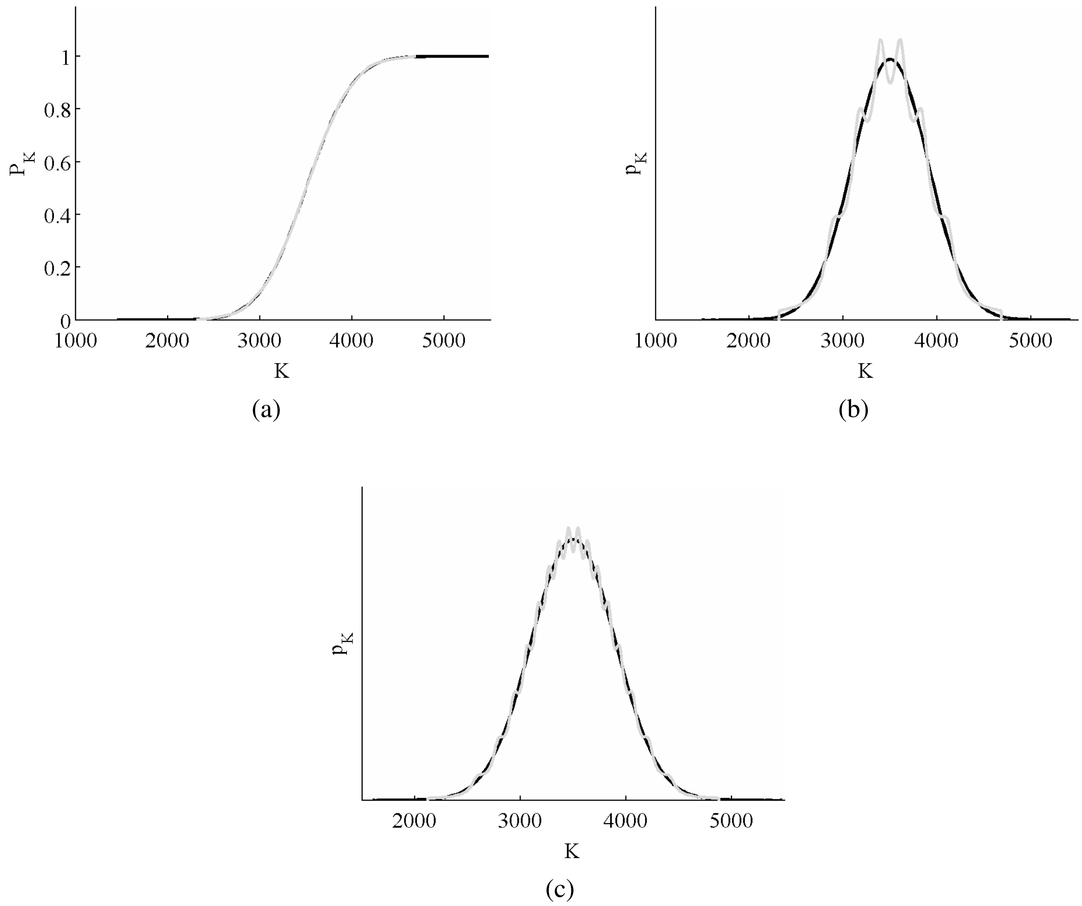


Figure 9: Comparison of empirical (a) CDF and (b) PDF obtained for $K$ with a 13th order expansion onto a Legendre basis; (c) is the PDF obtained for $K$ with a 34th order expansion onto a Legendre basis; black curves are the reference, grey curves are the PC representation.

where:

$$\left\{ \left\langle \mathbf{\Psi}_L^{\mathrm{T}} \left\{ \mathbf{\Psi}_H \right\}_p \mathbf{\Psi}_L \right\rangle \right\}_{rq} = \left\langle \psi_r^L \psi_p^H \psi_q^L \right\rangle = \sum_{i=1}^{n_{GL}} \psi_r^L(\zeta_i)\, \psi_p^H\left(\mathcal{T}\left(\zeta_i\right)\right)\, \psi_q^L(\zeta_i)\, w_i \tag{36}$$

In this way, results of the non intrusive approach are recovered with the intrusive one. Notice however that such a way leads to a higher numerical cost than the classical – one single basis – non-intrusive approach since the quadrature formula necessitates more integration points while more non-zeros terms stay within matrices.

However, for the current problem, although this two bases approach leads now to a satisfactory result, in contrast to the classical one with the Hermite basis, requiring such a high order is not a very efficient PC representation. Enrichment strategy of Ghosh and Ghanem (2008) seems an attractive way, since it is complementary to the current formulation. When using the same enrichment than that are proposed in Ghosh and Ghanem (2008) for our SDoF application, slightly improved results can be found but it is not tremendous.

More problematic is the fact that the Legendre basis is found to be less effective for expanding $U_\omega$ for low or high frequencies or a lower normalized coefficient of variation $c_\eta$ . In practice, it would not be desirable to change the expansion basis to cover all the frequency range of interest, since it becomes very difficult to manage it when a multi degrees of freedom system will be concerned. Hence, a compound basis would be a preferable solution, at the expense of an increase of the number of terms in the representation chosen, since it is more important to cover all the range of frequencies. It is the subject of the next subsection.

#### 4.2.3 Formulation of a mixed basis representation

From the results of the previous subsection, our proposal is to have a compound expansion basis made from the mix of bases that has been found to be effective in all the range of frequencies for the random variable. Hybrid formulation is not new since it is found in literature for SRBM based on stochastic Krylov subspace (see Sachdeva et al. (2006)). But the strict application of SRBM does not help for the application concerning a SDoF system. Indeed, the complete basis of Krylov subspace has a fixed dimension given by the number $m$ of degrees of freedom of the mechanical system. It is $m = 1$ in our SDoF system. Without conditioning, the first vector of the Krylov subspace is made from the force vector which is $f = 1$ here. However, we state that it is interesting to adapt the idea of reference Sachdeva et al. (2006) which combines Krylov subspace with PC expansion. To do this, we follow a similar idea of using Krylov subspace but paying no attention to the fixed limit for the number of vector in the basis.

Let us describe the strategy for the static case first. In a first step, the basis is supposed to be formed from the vectors sequence given by:

$$\mathbf{\Psi}_K = \left[ f,\, Kf,\, K^2 f,\, ...,\, K^{n_q-1} f \right] \tag{37}$$

where $n_q$ is thus a variable to be chosen. Interest here resides in the fact that the span of this basis would be included in the representation of the stochastic response $U_\omega$ to lead to:

$$U_\omega = \mathbf{\Psi}_K \mathbf{U} = \sum_{q=0}^{n_q-1} \left\{ \mathbf{\Psi}_K \right\}_q \left\{ \mathbf{U} \right\}_q \tag{38}$$

where the coefficients collected in the vector $\mathbf{U}$ are *a priori* random variables. Next, an hybrid formulation is obtained by representing each coefficient of this representation along a free basis $\boldsymbol{\Psi}$ of the gPC, saying, for instance, the Legendre basis $\boldsymbol{\Psi}_L$ . This leads to express $\mathbf{U}$ as:

$$\{\mathbf{U}\}_q = \boldsymbol{\Psi}_L \mathbf{u}_q = \sum_{r=0}^{n_r-1} \{\boldsymbol{\Psi}_L\}_r \{\mathbf{u}_q\}_r \tag{39}$$

to give:

$$U_\omega = \boldsymbol{\Psi}_{KL}\mathbf{u} \tag{40}$$

using $n_q \times n_r$ coefficients, such that $\mathbf{u}^{\mathrm{T}} = \left\{ \{\mathbf{u}_0\}_0 , \cdots , \{\mathbf{u}_0\}_{n_r} , \{\mathbf{u}_1\}_0 , \cdots , \{\mathbf{u}_1\}_{n_r} , \cdots \right\}$ for the basis $\boldsymbol{\Psi}_{KL} = \{ \{\boldsymbol{\Psi}_K\}_0 \{\boldsymbol{\Psi}_L\}_0 , \cdots , \{\boldsymbol{\Psi}_K\}_0 \{\boldsymbol{\Psi}_L\}_{n_r} , \{\boldsymbol{\Psi}_K\}_1 \{\boldsymbol{\Psi}_L\}_0 , \cdots , \{\boldsymbol{\Psi}_K\}_1 \{\boldsymbol{\Psi}_L\}_{n_r} , \cdots \}$.

The major consequence of choosing an infinitely expandable Krylov subspace is the non uniqueness of the basis coefficients collected in vector $\mathbf{U}$. To test the performance of the Krylov subspace having a constant participation factor, one can choose $n_r = 1$. On the contrary, the performance of the PC expansion only can be studied by choosing $n_q = 1$. But an interesting feature arising here is the mixing of bases. Each vector of bases $\boldsymbol{\Psi}_K$ and $\boldsymbol{\Psi}_L$ takes place in the composed resulting basis and they are terms such that $\{\boldsymbol{\Psi}_K\}_q \{\boldsymbol{\Psi}_L\}_r$. For instance, a two terms representation onto the Legendre basis means that Krylov basis coefficients follows a uniform law. Hence, the mixed basis introduced here can be viewed as the result of an enriched compound basis. This is equivalent to using an augmented set of basis vectors.

To adapt this strategy to dynamic systems, the static stiffness $K$ included in $\boldsymbol{\Psi}_K$ can be replaced by the dynamic one, being $\boldsymbol{\Psi}_K = [1,\, Z,\, Z^2,\, ...,\, Z^{n_q-1}]$ with $Z(\omega) = (K - \omega^2 + jc\omega)$, leading to complex basis vectors. For the numerical application, we have implemented this strategy in an intrusive approach using an importance sampling Monte Carlo method to evaluate moments using $10^6$ sample size, and a Gram-Schmidt procedure to orthonormalize the basis.

For the SDoF application, considering all the range of frequencies we have chosen to fix the orders such that $n_q = 2$ and $n_r = 13$ for the low and medium normalized coefficients of variations $c_\eta$ while it is $n_q = 12$ and $n_r = 12$ for the high normalized coefficient of variation $c_\eta$ . To assess the global efficiency of this strategy, due to the mixing of bases, it is not possible to report the order of truncation for a specified criterion, as it is done in the Figures 5 and 7. However, for low and high frequencies, it is found that coefficients collected in the vector $\mathbf{U}$ are almost constants, since the stochastic representation can be truncated at only one, single term. In contrast, for the resonant frequency of the nominal system, it is found that coefficients collected in the vector $\mathbf{U}$ are stochastics, by having non negligible values for all terms of $\mathbf{u}_q$. In Figure 10, we have thus chosen to report the number of non negligible values found in $\mathbf{u}$, being the ones that are greater than $10^{-7}$ (except for the order 0 term, the maximum coefficient is about $10^{-5}$). However, care should be taken in reading values of the proposed graph since they are dependent of the *a priori* choices made for $n_q$ and $n_r$, due to the non uniqueness of this kind of representation.

#### 4.2.4 Multi-element basis

A last test is performed which uses a piecewise expansion with low degree rather than a "one piece" expansion with high polynomial degree. As mentioned in the introduction, the idea was introduced a few years ago by Wan and Karniadakis (2005, 2006) and successfully applied to several mechanical problems (Le Meitour et al., 2010; Sarrouy et al., 2013b).

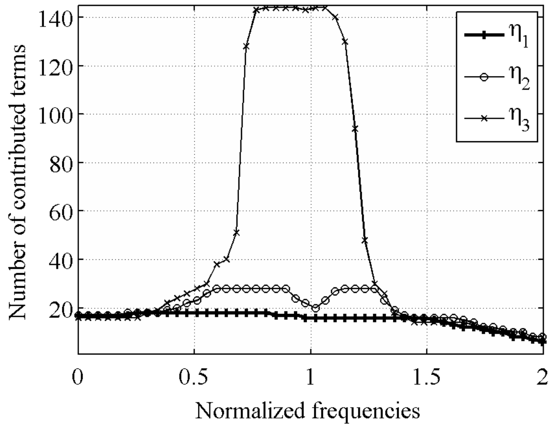


Figure 10: Number of contributing terms versus normalized frequencies for the Krylov-Legendre polynomial representation of the SDoF response when considering the three damping ratios $\eta_m$, $m \in \{1,\,2,\,3\}$.

Considering a normal random variable to describe the random input $K$, an isoprobabilistic tansformation is once again used to work with a uniform random variable $\zeta$ which has a bounded range of values $\mathcal{I} = [-1, 1]$. This range is easy to cut into pieces, that is to be partitioned into $n_e$ elements denoted $\mathcal{I}_n$. Over each element, a polynomial expansion with low degree is performed – either in an intrusive or a non intrusive way – using a Legendre basis in a local $\zeta_n$ random variable. This local variable $\zeta_n$ follows a uniform law over $[-1, 1]$ and is linked to $\zeta$ via an affine transformation $\zeta_n = \mathcal{T}_n(\zeta)$ such that $\mathcal{T}_n(\inf[I_n]) = -1$ and $\mathcal{T}_n(\sup[\mathcal{I}_n]) = +1$.

An important question is the choice of $\mathcal{I}_n$ elements. Adaptative strategies can be implemented to recursively refine the partition based on a given criterion (Wan and Karniadakis, 2005; Sarrouy et al., 2013a). Here, the partition will be set *a priori* and the quality of the global expansion will be measured by $\epsilon_{\text{CDF}}$. The obvious partition consists in $n_e$ elements with same size $2/n_e$. This gives no good results as due to the isoprobabilistic transformation, side elements $[-1, -1+2/n_e]$ and $[1-2/n_e, 1]$ account for "large" – indeed infinite – elements in $\xi$. Hence, the Gaussian tails are poorly represented. The chosen partition is then built to get $n_e - 2$ elements $I_n$, $2 \leq n \leq n_e - 1$ with an equal size in $\xi$: $\mathcal{T}(\sup[\mathcal{I}_n]) - \mathcal{T}(\inf[\mathcal{I}_n]) = 8/(n_e - 2)$ for $2 \leq n \leq n_e - 1$ and $\mathcal{T}(\inf[\mathcal{I}_2]) = -4$. The last two elements $\mathcal{I}_1$ and $\mathcal{I}_{n_e}$ complete the partition. Results are based on a non intrusive evaluation of the expansions over each element. To measure the accuracy of the global expansion, another criterion is defined:

$$\epsilon^2_{\text{PDF}}\left(U_\omega\right) = \frac{\int_{\inf[U_\omega]}^{\sup[U_\omega]} \left(\widehat{p}_{|U_\omega|}\left(u\right) - p_{|U_\omega|}\left(u\right)\right)^2 \mathrm{d}u}{\int_{\inf[U_\omega]}^{\sup[U_\omega]} \left(p_{|U_\omega|}\left(u\right)\right)^2 \mathrm{d}u} \tag{41}$$

where $\widehat{p}_{|U_\omega|}\left(u\right)$ denotes the empirical PDF of the distribution obtained from the MEgPC expansion, while $p_{|U_\omega|}\left(u\right)$ is the reference PDF (obtained via MCS using 1 million realizations).

Figure 11 shows the minimal number of elements required to satisfy either $\epsilon_{\text{PDF}} \leq 1\ \%$ criterion or $\epsilon_{\text{CDF}} \leq 1\ \%$ using a degree 3 expansion over each element. In this case, the fitted process is $|U_\omega|$ directly. Comparing the two families of curves, one can see that $\epsilon_{\text{PDF}} \leq 1\ \%$ is harder to satisfy than $\epsilon_{\text{CDF}} \leq 1\ \%$. Indeed, $\epsilon_{\text{CDF}}$ is not as sensitive as $\epsilon_{\text{PDF}}$ when discontinuities occur in the empirical PDF. For a unitary nominal frequency, 96 elements are required. The computational

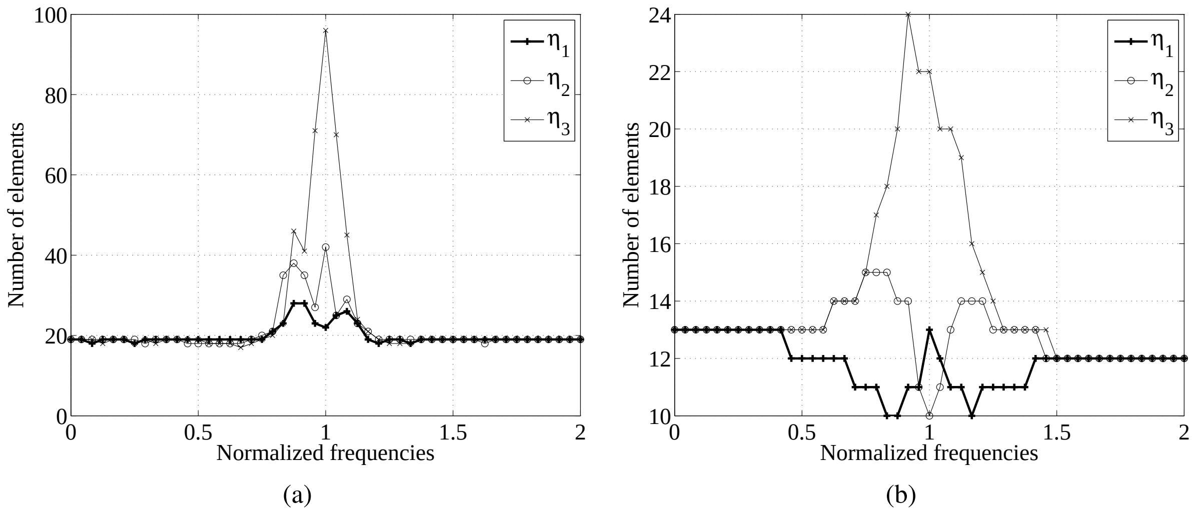


Figure 11: Number of elements required versus normalized frequencies for the MEgPC representation of the SDoF response when considering the three damping ratios $\eta_m$, $m \in \{1, 2, 3\}$ using (a) $\epsilon_{\mathrm{PDF}} \leq 1\%$ criterion and (b) $\epsilon_{\mathrm{CDF}} \leq 1\%$ criterion.

effort here is proportional to the number of elements: $n_e \times 4$ coefficients have to be evaluated (using a 3rd order expansion). Compared to a classical "one-piece" PC expansion, it means that a collection of $n_e$ small systems have to be solved in this case instead of a large system. In the case of a non intrusive evaluation of the coefficients, having low degrees over each element means that fewer points are required for the quadrature, but these values at quadrature nodes have to be evaluated for each element. Most importantly, it preserves low degrees for the expansion and thus prevents from bad conditioning and Gibbs phenomenon.

Figure 12 describes the CDF and PDF of amplitude $|U_\omega|$ at the natural frequency of the nominal system for $\eta_3$. 96 elements are required to satisfy the accuracy criterion based on $\epsilon_{\mathrm{PDF}}$. As showed by the figure, the CDF and the PDF returned by the expansion perfectly fit the reference curves. Finally, Figure 13 describes the PDFs of real and imaginary part of $U_\omega$ at the natural frequency of the nominal system for $\eta_3$ when fitting the complex process $U_\omega$ rather than the real valued process $|U_\omega|$. In this case, up to 116 elements are required to satisfy $\epsilon_{\mathrm{PDF}} \leq 1$ %.

## 5 CONCLUSION

While many studies focus on the handling of finite element models which have an increasing complexity as well as a large number of kinematic degrees of freedom and a large number of random variables, we focus here on the quality of the stochastic representation for the most simple, linear, system having one single degree of freedom with only one random variable. The stiffness is the input random variable, while the system response is the random ouput of the problem. The simplicity of the chosen situation enables the access with high fidelity to the expected result in order to gain insight to the PC expansion approach.

First of all, it is shown that response distribution functions strongly depend on a normalized coefficient of variation, being defined as the coefficient of variation of the random stiffness variable divided by the modal damping ratio of the nominal mechanical system. Then, for the experimentation, it is possible to fix the damping and varied the coefficient of variation and vice versa.

At the contrary of references Xiu and Karniadakis (2002); Xiu et al. (2003), we do not think

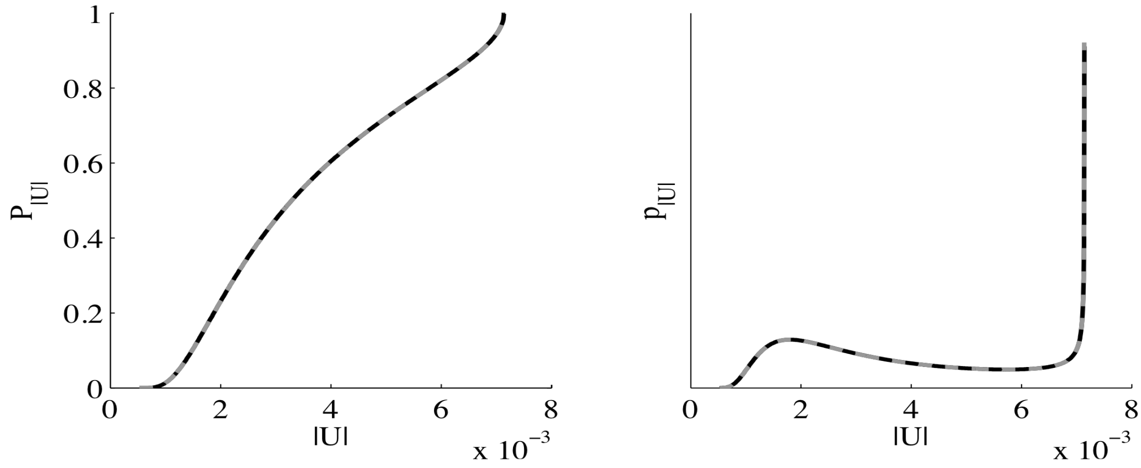


Figure 12: Comparison of empirical CDF and PDF module obtained for $U_\omega$ with a MEgPC representation using 96 elements and a 3rd order expansion over each element at the natural frequency of the nominal system for $\eta_3$ using an intrusive formulation; black curves are the reference, grey curves are the MEgPC representation. Error values are $\epsilon_{\text{CDF}} = 0.0051$ % and $\epsilon_{\text{PDF}} = 0.97$ %.

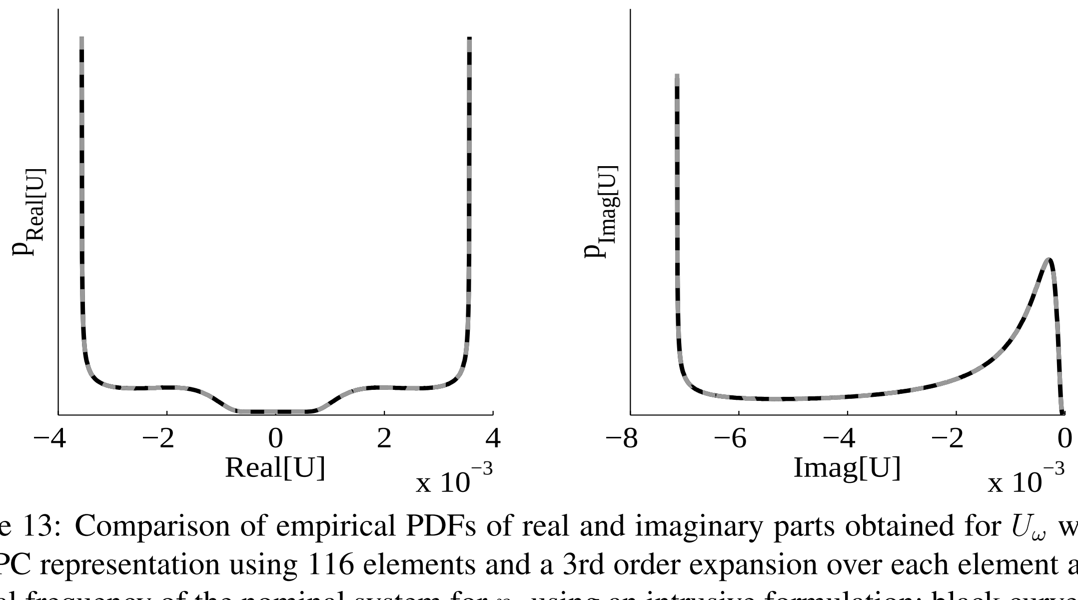


Figure 13: Comparison of empirical PDFs of real and imaginary parts obtained for $U_\omega$ with a MEgPC representation using 116 elements and a 3rd order expansion over each element at the natural frequency of the nominal system for $\eta_3$ using an intrusive formulation; black curves are the reference, grey curves are the MEgPC representation. Error values are $\epsilon_{\text{PDF}} = 0.83$ % for $\text{Real}[U]$ and $\epsilon_{\text{PDF}} = 0.93$ % for $\text{Imag}[U]$.

that the optimal basis for the input variable is always optimal for the output variable. It depends on the degree of non linearity between random variables, and assertion of Xiu and Karniadakis (2002); Xiu et al. (2003) becomes true only when approaching a linearity between them. For dynamical systems, it appears that stochastic responses become highly non-linear when reaching the resonant frequencies for a moderate to high normalized coefficient of variation. In these conditions, the representation of the output random variable can be very difficult when using the basis chosen for the input variable, as it is done for the standard PC approach.

Hence, a first proposed remedy to keep low the orders of expansion would be the use of two bases, one for the input variable and one other for the output variable. But strategies to handle two bases can be multiples, and this leads us to compound bases. It combines Krylov subspace with standard orthogonal polynomials. The compound bases approach proposed in this work has some similarity with an existing literature technique, although it is introduced here for different reasons, with other usage conditions. It seems to be particularly interesting since it would be always adapted to the investigated situation as it uses the input, known, random variable for the representation of the system output. In addition, the multi-element PC expansion is another way that is investigated. It is a robust approach in the sense that it can handle every situations without numerical difficulties since expansion order can be kept low by adding elements with smaller size. Both these approaches are appealing to tackle dynamic responses of SDoF systems.

For SDoF systems, the numerical cost involved by PC expansion is not applicable since MCS is very efficient in this situation, so it is not discussed in this work. The purpose of the study is rather to investigate the effectiveness of the stochastic bases for the representation of the dynamic response, hoping results found here will be useful for more general situations, involving multipledegrees of freedom stochastics systems.

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