Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXII, págs. 729-746 (artículo completo)
Carlos G. García Garino, Aníbal E. Mirasso, Mario A. Storti, Miguel E. Tornello (Eds.)
Mendoza, Argentina, 19-22 Noviembre 2013

ANALYSIS OF PARAMETRIC AND NON-PARAMETRIC UNCERTAINTIES IN THE DYNAMICS OF COMPOSITE THIN WALLED CURVED BEAMS

Marcelo T. Piovan[a,c] **and Rubens Sampaio**[a,b]

[a]*Centro de Investigaciones en Mecánica Teórica y Aplicada, Universidad Tecnológica Nacional - F.R.B.B., 11 de Abril 461, Bahá Blanca, BA, B8000LMI, Argentina, mpiovan@frbb.utn.edu.ar, http://www.frbb.utn.edu.ar*

[b]*Departament of Mechanical Engineering. Pontifícia Universidade Católica do Rio de Janeiro. Rua Marquês de São Vicente 225, Rio de Janeiro RJ-22453-90, Brazil, rsampaio@puc-rio.br, http://www.unmeseta.edu.ar/gia*

[c]*Consejo Nacional de Investigaciones Científicas y Tecnológicas (CONICET)*

Keywords: Uncertainties quantification, composite curved beams, dynamics.

Abstract. This article is devoted to the dynamic analysis of slender initially curved structures constructed with fiber reinforced composite materials. There are many ways to manufacture a composite material for uses in structural constructions, for example filament winding and resin transfer molding, among others. Depending on the manufacturing process composite materials may have deviations with respect to the calculated response (or deterministic response). These manufacturing aspects lead to uncertainty in the structural response associated with constituent proportions or geometric parameters among others. Another focus of uncertainty can be the mathematical model that represents the mechanics of the slender structure. In many structural models, the type of hypotheses invoked reflect the most relevant aspects of the physics of a structure, however in some circumstances these hypotheses are not enough, and cannot represent properly the mechanics of the structure. Uncertainties should be considered in a structural system in order to improve the predictability of a given modeling scheme. There are two strategies to face the uncertainties in the dynamics of structures: The parametric probabilistic approach and the non-parametric probabilistic approach. The first is related to quantify the uncertainty of given parameters such as variation of the angles of fiber reinforcement, material constituents, etc. The second is related to the uncertainty of the model which implies to consider uncertain the matrices of the whole system. In this study a shear deformable model of composite curved thin walled beams is employed as the mean model. The probabilistic model is constructed by adopting random variables for the uncertain entities (parameters or matrices) of the model. The probability density functions of the random variables are derived appealing to the Maximum Entropy Principle under given constraints. Once the probabilistic model is discretized in the context of the finite element method, the Monte Carlo method is employed to perform the simulations. Then the statistics of the simulations is evaluated and the parametric and non-parametric approaches are compared. Finally recommendations are outlined in the conclusion section.

1 INTRODUCTION

The employment of composite structures in different industrial devices is raising in the present years due to the outstanding features that the composite materials can offer, for example: high strength and stiffness properties together with a low weight, good corrosion resistance, enhanced fatigue life, low thermal expansion properties among others (Barbero, 1999). Other important feature of composite materials is the very low machining cost for complex structures (Jones, 1999). Slender composites structures that can be analyzed by means of curved beam models are present in many applications such as bridge segments, reinforcement of composite pressure vessel, machine parts (e.g. leaf springs of sport car), among others.

Since the eighties many research activities have been focused in the development of theoretical and computational methods for the dynamic and static analysis of slender composite structures, thin walled structures among them. Thus, the first consistent study dealing with the thin-walled composite-orthotropic members, was due to Bauld and Tzeng (1984), who developed a beam theory to analyze fiber-reinforced members featuring open cross-sections with symmetric laminates invoking Vlasov's hypotheses. Bauchau (1985) incorporated some aspects of shear flexibility in the analysis of thin-walled composite beams. The nineties brought a broad range of contribution in the analysis of thin walled composite beams that covered new theories for micro/macrostructures of composite materials, new modeling schemes including selective warping and second order displacements, etc. The works of Librescu and Song (1992), Song and Librescu (1993), Kim and White (1997) and Cesnik et al. (1996) are a few of the most representative works in the modeling of composite beams with thin or thick walled cross-sections; however most of them were devoted to closed cross-sections as basic models for the analysis of helicopter blades. More recently Cortínez and Piovan (2002) developed a theory of thin walled composite beams accounting for full shear flexibility, which means the consideration of shear deformation due to bending as well as due to warping related to non-uniform torsion. Piovan and Cortínez (2007a) extended the scopes and limits of the previous full shear flexible modeling conception by incorporating elastic couplings and the evaluation of general dynamic problems. In the work of Piovan and Cortínez (2007b) a curved thin walled composite beam theory was introduced that contains all the previous models as particular cases.

The behavior of composite structures under typical service in civil, aeronautical, aero-spatial or mechanical devices, is subjected to a number of factors that are stochastic in essence (Sriramula and Chryssantopoulos, 2009). Many researchers have focused their attention in the evaluation of the stochastic response of composite structures since the middles nineties Vickenroy and Wilde (1995); Salim et al. (1993). Moreover, there is a constant interest to quantify the propagation of uncertainty in the mechanics of composite materials at the microscale level (Sriramula and Chryssantopoulos, 2009) or for failure analysis (Pawar, 2011). The uncertainty involved in the material properties of the composites can be considered as random fields according to the works Mehrez et al. (2012b) and Mehrez et al. (2012a) among others. However, there are other ways for studying the dynamic response due to uncertainties in composite structures, for example by associating random variables to given entities that define a structural dynamic model. Effectively, when the parameters, such as material properties or reinforcement angles, are considered uncertain, this is called parametric probabilistic approach (PPA). However if the model as a whole is uncertain, this is called systemic uncertainty. In order to analyze this type of uncertainty there are various approaches, one of them is the so-called non-parametric probabilistic approach (NPPA). The last one implies the introduction of random matrix variables. This approach was formulated by Soize (2003) and employed in a variety of structural problems (Sampaio and Cataldo, 2011; Ritto et al., 2008).

In this article, the aforementioned probabilistic approaches are applied in order to evaluate the uncertainty in the dynamic response of naturally curved composite thin walled beams. The theory of the curved structure introduced by Piovan and Cortínez (2007b) and Piovan (2003) for several problems of structural mechanics is employed here as the deterministic model or the mean model. The solution of the dynamics equations is approximated in the context of the finite element method. For the PPA case, the parameters corresponding to elastic properties are considered uncertain. For the NPPA the stiffness matrix and the damping matrix are considered uncertain. This is due to the evidence gathered in other work of the authors (Piovan et al., 2013) in which the elastic properties, and hence the stiffness matrix, are the main focus of uncertainty propagation in dynamics of composite thin walled straight beams. To construct the probabilistic models, the probability density functions associated with the random variables are constructed based on the Maximum Entropy Principle (Jaynes, 1957a,b). This principle uses the available information of the uncertain entities to construct their probability density functions such that the Entropy in the sense of Shannon (1948) is maximum. The use of this scheme allows the maximum possible propagation of the uncertainty according to the available information about the random variables.

The article is organized as follows: after the introductory section where the state-of-the-art in modeling curved thin walled composite beams is summarized, the deterministic/mean model and its finite element discretization are briefly described, then the probabilistic approach is constructed. The parametric and the non-parametric approaches are described for this problem and the subsequent section contains the computational studies, the analysis of the uncertainty propagation in the dynamics of thin walled composite curved beams and finally concluding remarks are outlined.

2 DETERMINISTIC MODEL

2.1 Brief description of the curved beam model

In Fig. 1 a basic sketch of the thin walled beam is shown, where it is possible to see the reference points $\boldsymbol{C}$ and $\boldsymbol{A}$. The principal reference point $\boldsymbol{C}$ is located at the geometric center of the cross-section, where the x-direction is tangent to the curved axis of the beam while y and z are the axes of the cross section, but not necessarily the principal axes of inertias. The secondary reference system, located at $\boldsymbol{A}$, is used to describe shell stresses and strains. The curved axis of the beam, that has constant radius R, is contained in the plane Ξ. The curved beam has an opening angle β and a circumferential length $L = R\beta$. The present curved beam theory is based on the following assumptions (Piovan and Cortínez, 2007b):

1) The cross-section contour is rigid in its own plane.

2) The radius of curvature at any point of the shell is neglected. This implies to consider the section shaped in a polygonal arrangement.

3) The warping function is normalized with respect to the principal reference point $\boldsymbol{C}$.

4) A general laminate stacking sequence for composite material is considered.

5) The material density is considered constant along the beam.

6) Stress and strain components are defined according to the secondary reference system in $\boldsymbol{A}$, and the representative stresses are σ_{xx}, σ_{xs} and σ_{xn}.

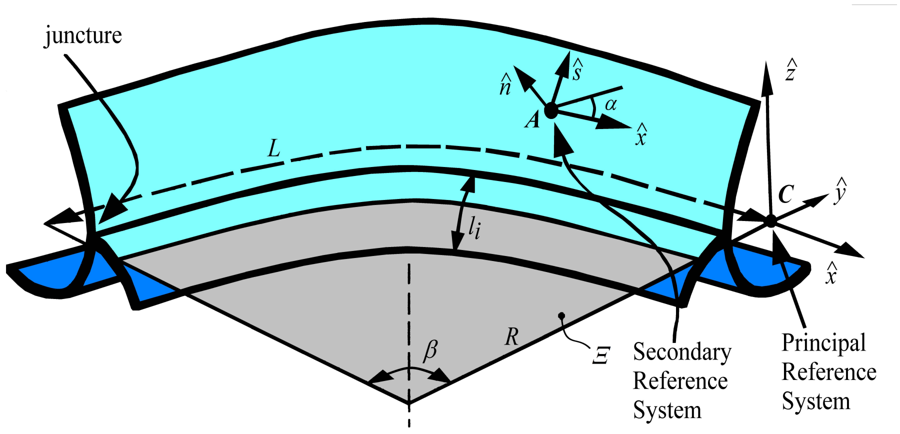


Figure 1: Sketch of the thin walled curved beam with the reference systems

7) The model is presented in the context of linear elasticity.

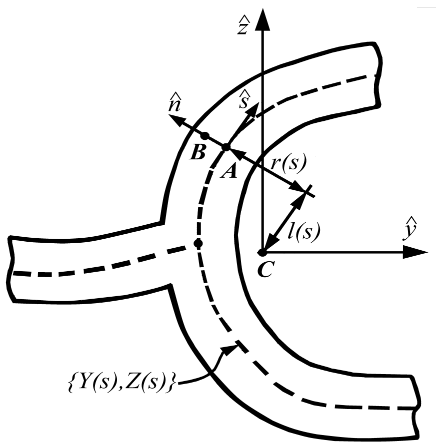


Figure 2: Sketch of the thin walled curved beam with the reference systems

Employing assumptions 1) to 7) one can derive the displacement field of the point ***B*** (Piovan and Cortínez, 2007b), which can be presented as follows:

$$\widetilde{\mathbf{U}}_{\boldsymbol{P}} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} u_{xc} - \omega\Phi_W \\ u_{yc} \\ u_{zc} \end{Bmatrix} + \begin{bmatrix} 0 & -\Phi_3 & \Phi_2 \\ \Phi_3 & 0 & -\Phi_1 \\ -\Phi_2 & \Phi_1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix}, \tag{1}$$

Where Φ_W, Φ_1, Φ_2 and Φ_3 are defined in terms of rotational and warping parameters as follows:

$$\Phi_1 = \phi_x, \quad \Phi_2 = \theta_y, \quad \Phi_3 = \theta_z - \frac{u_{xc}}{R}, \quad \Phi_W = \theta_x + \frac{\theta_y}{R} \tag{2}$$

and u_{xc}, u_{yc}, u_{zc} are the displacements of the reference center in x-, y-, and z- directions, respectively. θ_z and θ_y are bending rotational parameters. ϕ_x is the twisting angle and θ_x is a warping-intensity parameter. R is the radius of curvature of the beam. In Eq. (1) the cross-sectional variables $y(s)$ and $z(s)$ of a generic point are related to the ones of the wall middle line $Y(s)$ and $Z(s)$ by means of Eq. (3) is the warping function normalized with respect to the reference center. It is defined in Eq. (4)

$$y(s) = Y(s) - n\frac{dZ}{ds},\ z(s) = Z(s) + n\frac{dY}{ds}, \tag{3}$$

$$\omega(s,n) = \omega_p(s) + \omega_s(s,n). \tag{4}$$

In Eq. (4), $\omega_p(s)$ is the primary or contour warping function whereas $\omega_s(s,n)$ is the secondary or thickness warping. These entities are given by:

$$\omega_p(s) = \int_s [r(s) + \psi(s)]\, ds - D_C,\ \omega_s(s,n) = -nl(s), \tag{5}$$

where the functions $r(s)$, $l(s)$, $\psi(s)$ and D_C are defined in the following form (see Fig. 2):

$$\begin{aligned} r(s) &= Z(s)\frac{dY}{ds} - Y(s)\frac{dZ}{ds}, l(s) = Y(s)\frac{dY}{ds} + Z(s)\frac{dZ}{ds}, \\ \psi(s) &= \frac{1}{\bar{A}_{66}(s)}\left[\frac{\int_s r(s)ds}{\oint_S \frac{1}{\bar{A}_{66}(s)}ds}\right], D_C = \frac{\oint_S [r(s) + \psi(s)]\,\bar{A}_{11}(s)ds}{\oint_S \bar{A}_{11}(s)ds}. \end{aligned} \tag{6}$$

The functions $\bar{A}_{11}$ and $\bar{A}_{66}$ are normal and tangential elastic properties of composite laminates (Piovan and Cortínez, 2007a) which can vary along the section middle line. $\psi(s)$ is function related to the torsional shear flow and D_C is a constant to normalize the warping function with respect to the reference system $\boldsymbol{C}$ (Cortínez and Piovan, 2002; Piovan and Cortínez, 2007b). In open sections, $\psi(s) = 0$, then Eq. (6) is valid for closed sections as well as for open sections. The warping function described in Eq. (4), has an analogous form to the ones defined by Song and Librescu (1993) or **?** for closed sections and straight beams.

The displacement-strain relations can be obtained by substituting Eq. (1) in the well-known expressions of linear strain components. As it was shown by Piovan and Cortínez (2007b) the shell strains can be written as:

$$\widetilde{\mathbf{E}}_P = \mathbf{G}_k\widetilde{\mathbf{D}}, \tag{7}$$

where:

$$\begin{aligned} \widetilde{\mathbf{E}}_P^T &= \{\epsilon_{xx}, \gamma_{xs}, \gamma_{xn}, \kappa_{xx}, \kappa_{xs}\}, \\ \widetilde{\mathbf{D}}^T &= \{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}, \varepsilon_{D5}, \varepsilon_{D6}, \varepsilon_{D7}, \varepsilon_{D8}\}, \end{aligned} \tag{8}$$

$$\mathbf{G}_k = \begin{bmatrix} 1 & Z & -Y & -\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {dY}/{ds} & {dZ}/{ds} & r(s)+\psi(s) & -\psi(s) \\ 0 & 0 & 0 & 0 & -{dZ}/{ds} & {dY}/{ds} & l(s) & 0 \\ 0 & -{dY}/{ds} & {dZ}/{ds} & -l(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{9}$$

In Eq. (8), ϵ_{xx}, γ_{xs} and γ_{xn} are the strain components and κ_{xx}, κ_{xs} are the curvatures of the shell that conforms the wall of the cross-section. These strain components are measured according to the wall reference system in $\boldsymbol{A}$. The entities ε_{Di}, $i = 1,...,8$ may be regarded as generalized deformations. In this context ε_{D1} is the axial deformation, ε_{D2} and ε_{D3} are bending deformations, ε_{D3} is the deformation due to non-uniform warping, ε_{D5} and ε_{D6} are the bending shear deformations, ε_{D7} is the warping shear deformation and finally ε_{D8} is the pure torsion shear deformation. These generalized deformations, which are collected in vector $\widetilde{\mathbf{D}}$, are defined in the following form:

$$\widetilde{\mathbf{D}} = \mathbf{G}_{DU}\widetilde{\mathbf{U}}, \tag{10}$$

where $\mathbf{G}_{DU}$ is a matrix operator and $\widetilde{\mathbf{U}}$ is the vector of kinematic variables which are defined in following forms, in which $\partial_x(\diamond)$ is the spatial derivative operator.

$$\mathbf{G}_{DU} = \begin{bmatrix} \partial_x(\diamond) & 1/R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_x(\diamond) & -1/R & 0 \\ -\partial_x(\diamond)/R & 0 & \partial_x(\diamond) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_x(\diamond)/R & 0 & \partial_x(\diamond) \\ 0 & \partial_x(\diamond) & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x(\diamond) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_x(\diamond) & -1 \\ 0 & 0 & 0 & 0 & 1/R & \partial_x(\diamond) & 0 \end{bmatrix}, \tag{11}$$

$$\widetilde{\mathbf{U}}^T = \{u_{xc}, u_{yc}, \theta_z, u_{zc}, \theta_y, \phi_x, \theta_x\}. \tag{12}$$

The principle of virtual works can be condensed in the following form:

$$\mathcal{W}_T = \int_L \left(\delta\widetilde{\mathbf{D}}^T\widetilde{\mathbf{Q}}\right) dx + \int_L \delta\widetilde{\mathbf{U}}^T\mathbf{M}_m\ddot{\widetilde{\mathbf{U}}}dx - \int_L \delta\widetilde{\mathbf{U}}^T\widetilde{\mathbf{P}}_X dx + \delta\widetilde{\mathbf{U}}^T\widetilde{\mathbf{B}}_X\Big|_{x=0}^{x=L} = 0, \tag{13}$$

where the force vector $\widetilde{\mathbf{Q}}$ is defined as follows:

$$\widetilde{\mathbf{Q}}^T = \{Q_x, M_y, M_z, B, Q_y, Q_z, T_w, T_{sv}\}, \tag{14}$$

whereas for the sake of fluid and clear reading, the matrix of mass coefficients $\mathbf{M}_m$, the vector $\tilde{\mathbf{P}}_X$ of external forces and the vector $\tilde{\mathbf{B}}_X$ of natural boundaries conditions are detailed in Appendix A. Q_x, M_y, M_z, and B identify the axial force, the bending moment in y-direction, the bending moment in z-direction, and the bi-moment, respectively; whereas Q_y, Q_z, T_w, and T_{sv} correspond to the shear force in y-direction, the shear force in z-direction, the twisting moment due to warping and the twisting moment due to pure torsion, respectively. These internal/generalized forces can be written in terms of the shell-forces as (Piovan and Cortínez, 2007a):

$$\widetilde{\mathbf{Q}} = \int_S \mathbf{G}_k^T\widetilde{\mathbf{N}}_P ds, \tag{15}$$

where $\widetilde{\mathbf{N}}_P$ is the vector of shell stress resultants or shell forces and moments defined according to (Barbero, 1999; Jones, 1999):

$$\widetilde{\mathbf{N}}_P^T = \int_S \{\sigma_{xx}, \sigma_{xs}, \sigma_{xn}, n\sigma_{xx}, n\sigma_{xs}\} dn. \tag{16}$$

The differential equations of motion and corresponding boundary conditions are derived by applying variational procedures in Eq. (13). The differential equations of motion can be useful for some numerical methods, e.g. power series method or differential quadrature. While in the present article the finite element method is employed, the derivation of differential equations is not necessary. The interested readers may follow in the technical literature authors' articles (Piovan and Cortínez, 2007b; Piovan, 2003) devoted to evaluate the differential equations of the thin-walled curved beam model applied to a number of specific structural problems.

2.2 Constitutive equations in terms of internal forces and generalized strains

In order to obtain the relationship between beam stress resultants and generalized deformations ε_{Di}, one has to select the constitutive laws for a composite shell and employ constitutive hypotheses (Piovan and Cortínez, 2007a) of the shell stress resultants in terms of the shell strains. The shell stress resultants can be expressed in terms of the generalized deformations defined in Eq. (10) in the following matrix form:

$$\widetilde{\mathbf{N}}_P = \mathbf{M}_C \widetilde{\mathbf{E}}_P, \tag{17}$$

where $\mathbf{M}_C$ is the matrix of modified shell stiffness, which depends on the type of constitutive hypotheses involved (Piovan, 2003) and can be expressed in the following form:

$$\mathbf{M}_C = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{16} & 0 & \bar{B}_{11} & \bar{B}_{16} \\ & \bar{A}_{66} & 0 & \bar{B}^*_{16} & \bar{B}_{66} \\ & & \bar{A}^*_{55} & 0 & 0 \\ & sym & & \bar{D}_{11} & \bar{D}_{16} \\ & & & & \bar{D}_{66} \end{bmatrix}. \tag{18}$$

Due to the lack of space the coefficients $\bar{A}_{11}$, $\bar{B}_{11}$, $\bar{D}_{11}$, etc, are not described in the present article, however the interested reader can found them in the works of Piovan and Cortínez (2007b) or Piovan (2003).

Substituting Eq. (17) into Eq. (15) the beam stress resultants can be obtained in terms of generalized strains:

$$\widetilde{\mathbf{Q}} = \mathbf{M}_k \widetilde{\mathbf{D}}, \tag{19}$$

where:

$$\mathbf{M}_k = \int_S \mathbf{G}_k^T \mathbf{M}_C \mathbf{G}_k ds. \tag{20}$$

The matrix $\mathbf{M}_k$ of cross-sectional stiffness coefficients, leads to constitutive elastic coupling or not, depending on the stacking sequence of the laminates in a given cross-section. The interested reader can follows extended explanation about elastic constitutive coupling in the books of Barbero (1999) and Jones (1999). Moreover for beam applications the explanation of the constitutive coupling can be followed in the works of Piovan and Cortínez (2007b) and Kim and White (1997) among others.

2.3 Finite element approach

A quartic order iso-parametric finite element of five nodes with seven degree of freedom per node is employed to solve the motion equations. The formulation of the finite element approach for this type of curved structural member has been introduced in the works of Piovan and Cortínez (2007b) and Piovan (2003), where the interested reader can find detailed explanations.

Thus, the finite element equation of the assembled system can be written in the conventional form as:

$$\mathbf{K}\bar{\mathbf{W}} + \mathbf{C}_{RD}\dot{\bar{\mathbf{W}}} + \mathbf{M}\ddot{\bar{\mathbf{W}}} = \bar{\mathbf{F}}, \tag{21}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global matrices of elastic stiffness and mass, respectively; whereas $\bar{\mathbf{W}}$, $\ddot{\bar{\mathbf{W}}}$ and $\bar{\mathbf{F}}$ are the global vectors of nodal displacements, nodal accelerations and nodal forces, respectively. $\mathbf{C}_{RD}$ is the global matrix of structural damping, calculated according to the Rayleigh's definition as:

$$\mathbf{C}_{RD} = \eta_1\mathbf{M} + \eta_2\mathbf{K}. \tag{22}$$

The coefficients η_1 and η_2 in Eq. (22) are computed by using the damping coefficients, ξ_1 and ξ_2, according to the common methodology presented in the bibliography related to finite element procedures (Bathe, 1996).

The response in the frequency domain of the linear dynamic system given by Eq. (21) can be written as:

$$\widehat{\mathbf{W}}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}_{RD} + \mathbf{K}\right]^{-1}\widehat{\mathbf{F}}(\omega), \tag{23}$$

where $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{F}}$ are the Fourier transform of the displacement vector and force vector, respectively; whereas ω is the circular frequency measured in [rad/sec].

3 DESCRIPTION OF THE PROBABILISTIC MODEL

The probabilistic model is constructed selecting parameters or matrices and associating the corresponding random variables based on the available information. Whether it is used PPA or NPPA, the probabilistic approach is constructed from the finite element equation of the deterministic model. The construction of the probability density functions of the random variables is quite sensitive in stochastic analysis and they should be deduced according to the given information (normally scarce) about the uncertain parameters. The Maximum Entropy Principle is a good strategy to select the probabilistic model despite the lack of experimental data. Thus, the Maximum Entropy Principle allows to derive the probability density functions of the random variables guaranteeing consistence with the available information and the physics of the problem.

In order to derive the probability density functions of the random variables, the Maximum Entropy Principle is proposed in the following form:

$$p_V^{(opt)} = \arg\max_{p_V\in\mathfrak{P}} S(p_V) \tag{24}$$

where $p_V^{(opt)}$ is the optimal probability density function such that $S(p_V^{(opt)}) \geq S(p_V), \forall p_V \in \mathfrak{P}$, and S is the measure of entropy whereas $\mathfrak{P}$ is a set of admissible probability density functions satisfying the known data of the random variables and the physical constraints. The measure of the entropy S is defined as **?**:

$$S(p_V) = -\int_{\mathfrak{S}} p_V \ln(p_V)\, dv \tag{25}$$

where $\mathfrak{S}$ is the support of the probability distributions of the random variables taken into account in the optimization procedure.

Once the random variables are appropriately defined then the stochastic finite element equation can be written, through Eq. (23), in the following form:

$$\widehat{\mathbb{W}}\left(\omega\right)=\left[-\omega^{2}\mathbf{M}+i\omega\mathbb{C}_{RD}+\mathbb{K}\right]^{-1}\widehat{\mathbf{F}}\left(\omega\right). \tag{26}$$

Notice that in Eq. (26) the math-blackboard typeface indicates stochastic entities, thus the stiffness matrix $\mathbb{K}$ is stochastic because random variables (scalars or matrices) are employed in its construction, and the damping matrix $\mathbb{C}_{RD}$ is stochastic through the stochastic nature of $\mathbb{K}$ in Eq. (22), hence $\widehat{\mathbb{W}}$ is stochastic.

The Monte Carlo method is used the simulate the stochastic dynamics. This strategy leads to the calculation of a deterministic system for each realization of the random variables employed. The convergence of the stochastic response $\widehat{\mathbb{W}}$ can be calculated with the following function:

$$conv\left(N_{MS}\right)=\sqrt{\frac{1}{N_{MS}}\sum_{j=1}^{N_{MS}}\int_{\Omega}\left\|\widehat{\mathbb{W}}_{j}\left(\omega\right)-\widehat{\mathbf{W}}\left(\omega\right)\right\|^{2}d\omega}, \tag{27}$$

where N_{MS} is the number of Monte Carlo samplings and Ω is the frequency band of analysis. Clearly, $\widehat{\mathbb{W}}$ is the response of the stochastic model and $\widehat{\mathbf{W}}$ the response of the mean model or deterministic model.

Due to the enormous calculation time involved in this type of studies, in Eq. (23) and/or Eq. (26) a modal decomposition is used in the calculation of the spectral range of interest. With this type of scheme the amount of time saved could be 10 to 20 times the conventional calculation time.

3.1 Parametric approach

The stochastic model according to the PPA is constructed selecting two sets of uncertain parameters and associating random variables to them. One set for the orientation angles of the fiber reinforcement in the layers of each panel and other set for basic elastic properties of the material. In the present problem random variables V_i, $i=1,2...N_P$ and V_i, $i=N_P+1,...,N_P+6$ are introduced such that they represent the angles of N_P different plies in a cross-sectional laminate and the basic elastic properties of the material (i.e. elastic moduli: E_{11}, $E_{22}=E_{33}$, $G_{12}=G_{13}$ and G_{23}, Poisson coefficients: $\nu_{12}=\nu_{13}$ and ν_{23}), respectively.

The available information to obtain the probability density functions is related to some information extracted from the technical literature (Sriramula and Chryssantopoulos, 2009). The following conditions are invoked in order to construct the probability density functions with the Maximum Entropy Principle:

- The random variables associated with material properties are positive and defined in bounded supports.
- The random variables associated with the reinforcement angles have bounded supports whose upper and lower limits are distant Δ_α from the expected value $\underline{V}_i$.
- The expected values are $\mathcal{E}\{V_i\}=\underline{V}_i$, $i=1,...N_P+6$, i.e. those corresponding to the deterministic model.
- The variance of the random variable has to be kept finite in order to satisfy the physical consistence of the problem.

- There is no information about the correlation between random variables.

Consequently, according to the aforementioned background, the probability density functions of the random variables V_i can be written as:

$$p_{V_i}(v_i) = \mathfrak{S}_{\left[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}\right]}(v_i) \frac{1}{2\Delta_\alpha}, i = 1, ..., N_P \tag{28}$$

$$p_{V_i}(v_i) = \mathfrak{S}_{\left[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}\right]}(v_i) \frac{1}{2\sqrt{3}\underline{V}_i \delta_{V_i}}, i = N_P + 1, ..., N_P + 6 \tag{29}$$

where $\mathfrak{S}_{\left[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}\right]}(v_i)$ is the generic support function, whereas $\mathcal{L}_{V_i}$ and $\mathcal{U}_{V_i}$ are the lower and upper bounds of the random variable V_i. Δ_α is a gap measured in angular units (radians or degrees), whereas δ_{V_i} is the coefficient of variation. The Matlab function unifrnd$(\underline{V}_i - \Delta_\alpha, \underline{V}_i + \Delta_\alpha)$ can be used to generate realizations of the random variables V_i, $i = 1, 2...N_P$. The Matlab function unifrnd$\left(\underline{V}_i\left(1 - \delta_{V_i}\sqrt{3}\right), \underline{V}_i\left(1 + \delta_{V_i}\sqrt{3}\right)\right)$ can be used to generate realizations of the random variables V_i, $i = N_P + 1, ..., N_P + 6$.

3.2 Non-parametric approach

Under this conception, the matrices of the system are considered uncertain. In particular, there is evidence (Piovan et al., 2013) that the uncertainty in elastic properties is more sensitive than the uncertainty in mass properties in the dynamics of beams constructed with composite materials. Consequently, the construction of the probability density function of the random stiffness matrix $\mathbf{K}$ is performed in this section. The procedure explained in the subsequent lines follows the concepts and ideas elaborated in the works of Soize (2001, 2003, 2005).

In order to construct the random matrix $\mathbb{K}$, it is necessary that the mean value (or the deterministic one) of the positive-definite matrix $\mathbf{K}$ could be written according to the Cholesky-decomposition, that is: $\mathbf{K} = \mathbf{L}_\mathbf{K}^T \mathbf{L}_\mathbf{K}$, where $\mathbf{L}_\mathbf{K}$ is an upper triangular matrix. Hence the random matrix $\mathbb{K}$ can be written as follows:

$$\mathbb{K} = \mathbf{L}_\mathbf{K}^T \mathbb{G} \mathbf{L}_\mathbf{K} \tag{30}$$

where $\mathbb{G}$ is a random matrix that has the following constraints:

- Positive-definiteness.
- The mean value is the identity matrix: $\mathcal{E}\{\mathbb{G}\} = \mathbf{I}$.
- The mean square value of its inverse is finite, i.e $\mathcal{E}\left\{\|\mathbb{G}^{-1}\|_F^2\right\} < +\infty$; this assures that the response of the system is a second-order random variable.

Then using the Maximum Entropy Principle the probability density function of $\mathbb{G}$ can be written as (Soize, 2001):

$$p_\mathbf{G}(\mathbf{G}) = \mathfrak{S}_{\mathbb{M}^+(\mathbb{R})}(G)\, C_\mathbb{G} \det(\mathbf{G})^{(n+1)\frac{1-\delta^2}{2\delta^2}} \exp\left\{-\frac{n+1}{2\delta^2} tr(\mathbf{G})\right\} \tag{31}$$

where $\mathfrak{S}_{\mathbb{M}^+(\mathbb{R})}(G)$ is the support of the random variable, n is the dimension of the random matrix $\mathbb{G}$, the dispersion parameter δ and $C_{\mathbb{G}}$ are given as follows:

$$\delta = \sqrt{\frac{1}{n}E\left\{\|\mathbb{G}-\mathbf{I}\|_F^2\right\}}, C_{\mathbb{G}} = \frac{(2\pi)^{\left(n-n^2\right)/4}\left(\frac{n+1}{2\delta^2}\right)^{\left(n+n^2\right)/\left(2\delta^2\right)}}{\prod_{j=1}^{n}\Gamma\left(\frac{n+1}{2\delta^2}+\frac{1-j}{2}\right)} \tag{32}$$

The dispersion parameter is such that $0<\delta<\sqrt{(n+1)/(n+5)}$.

Thus, for each realization of the random matrix $\mathbb{K}$, the matrix $\mathbb{G}$ is built by means of a Cholesky decomposition, i.e. $\mathbb{G}=\mathbb{L}^T\mathbb{L}$, where $\mathbb{L}$ is an upper triangular positive-definite random matrix subjected to the following constraints:

- The random variables $\{\mathbb{L}_{jk}, j\le k\}$ are independent.
- For $j<k$, the real-valued random variable $\mathbb{L}_{jk}=\sigma V_{jk}$, in which $\sigma=\delta\sqrt{n+1}$ and V_{jk} is a real-valued random variable with zero mean and unit variance.
- For $j=k$ the real-valued random variable $\mathbb{L}_{jk}=\sigma\sqrt{2V_j}$, in which V_j is a real-valued gamma random variable with probability density function:

$$p_{V_j}(v) = \mathfrak{S}_{\mathbb{R}^+}(v)\frac{v^{\left(\frac{n+1}{2\delta^2}-\frac{1-j}{2}\right)}}{\Gamma\left(\frac{n+1}{2\delta^2}+\frac{1-j}{2}\right)}\exp(v) \tag{33}$$

As it is possible to infer, the random variables $V_{jk}, j\neq k$ and $V_{jk}, j=k$ can be generated by a normal distribution and a gamma distribution respectively. In fact they can be generated in the Monte Carlo simulation procedure by means of the Matlab functions $normrnd(0,1)$ and $gamrnd(\alpha,\beta)$, with $\alpha=\left(\frac{n+1}{2\delta^2}+\frac{1-j}{2}\right)$ and $\beta=1$.

4 COMPUTATIONAL STUDIES

In this section a study is carried out related to the propagation of uncertainties due to material properties and/or constructive aspects of composite laminates, in the dynamic response of curved thin-walled composite beams. For this study a clamped-free beam (length $L=6.0\ m$, radius $R=6.0\ m$) with rectangular cross-section is employed. The following Fig. 3 shows the rectangular cross-section with the secondary reference systems associated to each panel. Moreover it is possible to see the excitation due to an impulsive unitary force located at $x,y,z=L,b/2,h/2$ and oriented with $\psi=45^o$. he web height and flange width are $h=0.6\ m$, $b=0.3\ m$, whereas the thickness of all laminates is $e=0.03\ m$. Each laminate is composed by 8 laminas of equal thickness. The material of the beams is graphite-epoxy (AS4/3501-6) whose properties are: $E_{11}=144\ GPa$, $E_{22}=E_{33}=9.68\ GPa$, $G_{12}=G_{13}=4.14\ GPa$, $G_{23}=3.45\ GPa$, $\nu_{12}=\nu_{13}=0.3$, $\nu_{23}=0.5$, and the density $\rho=1389\ Kg/m^3$. Although the damping coefficients could be uncertain, in this study they assume fixed values $\xi_1=0.05$ and $\xi_2=0.05$ in order to facilitate the analysis of uncertainty connected with elastic properties and the modeling itself.

The stacking sequences to be used are described in Table 1, in which the acronyms CUS and CAS stand for "Circumferential Uniform Stiffness" and "Circumferential Asymmetric Stiffness". These acronyms were introduced by Rehfield et al. (1990) to identify the type of lamination scheme for rectangular cross-sections. The CUS laminate involves elastic constitutive coupling between twisting moments and axial force as well as both shear forces and both bending moments, whereas the CAS laminate involves elastic constitutive coupling between bending

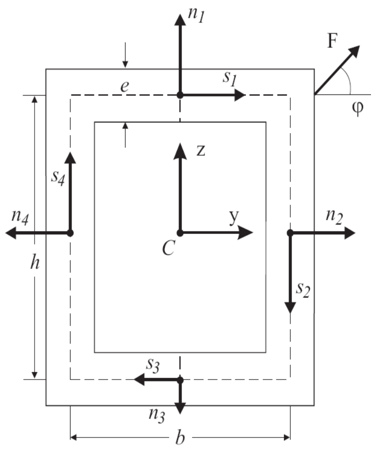


Figure 3: Rectangular cross-section with reference systems

moments and twisting moments together with coupling of the axial force with both shear forces (Piovan and Cortínez, 2007a; Piovan, 2003; Rehfield et al., 1990).

Cross-section	Laminate Name	Angle orientation
□	CUS(α)	left and right panels: $\{(\alpha, \alpha)_4\}$
		upper and lower panels: $\{(\alpha, \alpha)_4\}$
	CAS(α)	upper and right panels: $\{(\alpha, \alpha)_4\}$
		lower and left panels: $\{(-\alpha, -\alpha)_4\}$

Table 1: Lamination schemes for the cross-sections.

The stochastic analysis is mainly concerned with the evaluation of the uncertainty propagation in the frequency response function of the composite beam subjected to a unit force F used to perturb the structure. The force is located at the free end of the beam ($x = L$) according to Fig 3. The response is observed at the free end, and it is evaluated by means of the following frequency response function:

$$H_F(\omega) = \frac{\left\|\widehat{\mathbf{U}}_{\boldsymbol{P}}(\omega)\right\|}{\widehat{F}(\omega)}. \tag{34}$$

In Eq. (34), $\left\|\widehat{\mathbf{U}}_{\boldsymbol{P}}\right\|$ is the norm of the Fourier transform of the displacement vector of the point (calculated according to Eq. (1)) where the force is applied (see Fig 3) and $\widehat{F}$ is the Fourier transform of the force applied at the beam's end. Moreover, other frequency response functions are introduced for specific comparative purposes, that is:

$$H_1(\omega) = \frac{\widehat{u}_{yc}(\omega)}{\widehat{F}_y(\omega)}, H_2(\omega) = \frac{\widehat{u}_{zc}(\omega)}{\widehat{F}_z(\omega)}, H_3(\omega) = \frac{\widehat{\phi}_x(\omega)}{\widehat{T}_x(\omega)}, \tag{35}$$

where $\widehat{u}_{yc}$, $\widehat{u}_{zc}$ and $\widehat{\phi}_x$ are the Fourier transforms of lateral displacement, vertical displacement and twisting angle, respectively, whereas $\widehat{F}_y$, $\widehat{F}_z$ and $\widehat{T}_x$ are the Fourier transforms of the components of force F and the associated twisting moment. For this problem, the displacements are calculated at the free end.

Models of twelve finite elements of five nodes are used for the deterministic and stochastic calculations (i.e. models of 336 degree of freedom). This number of elements is enough to assure Piovan (2003) a precision of more than 99% up to the eighth natural frequency. Moreover a modal decomposition, with up to 24 modal coordinates, has been employed in order to reduce the calculation procedure.

For the PPA, four random variables are selected for the orientation angles of the fibre reinforcement according to the common stacking sequences employed in the construction of composite structures. These random variables have the following expected values: $\mathcal{E}\{V_1\} = 0^o$, $\mathcal{E}\{V_2\} = 15^o$, $\mathcal{E}\{V_3\} = 45^o$ and $\mathcal{E}\{V_4\} = 90^o$, with $\Delta_\alpha = 2^0$. On the other hand the expected values of random variables V_i, $i = 5, ..., 10$ correspond to the nominal values of the elastic properties indicated above. The elastic random variables can have dispersion parameters contained in $\delta_i \in [0.04, 0.12]$, $i = 5, ..., 10$ (Sriramula and Chryssantopoulos, 2009; Piovan et al., 2013). In the case of the NPPA it is important to identify the limits of the dispersion parameter that according to section 3.2 it should be: $0 < \delta_{\mathbb{K}} < \sqrt{(336+1)/(336+5)} = 0.9927$. With the scope of analyzing the effect of the uncertainty related to the whole model, the following set of dispersion parameters $\delta_{\mathbb{B}} \in [0.35, 0.50, 0.65, 0.80, 0.95]$ is used in the simulations.

The Fig 4 shows an example of the convergence of the Monte Carlo simulations by studying the evolution of the function $conv\,(N_{MS})$ with respect to the number of simulations in the whole range of the 500 realizations. In these cases it is employed a cross-section with a CAS(45) lamination sequence such that $\Delta_\alpha = 2^o$ and $\delta_i = 0.1, i = 1, ..., N_P + 6$ for the PPA and $\delta_{\mathbb{K}} = 0.8$ for the NPPA. It can be seen that with nearly 250 simulations, the $conv\,(N_{MS})$ function reaches an acceptable level convergence. The convergence analysis has been performed in every simulation giving similar results.

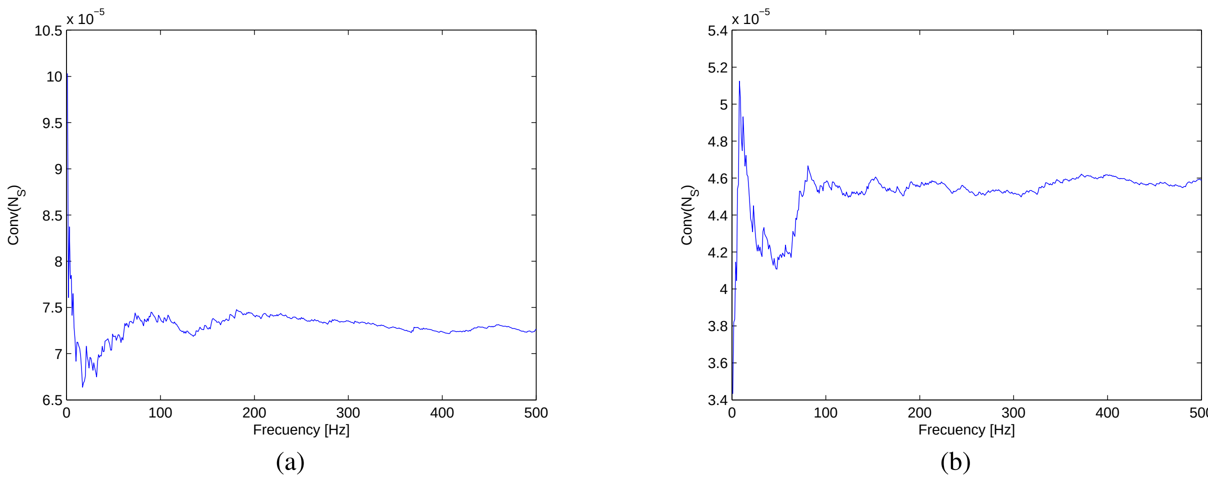
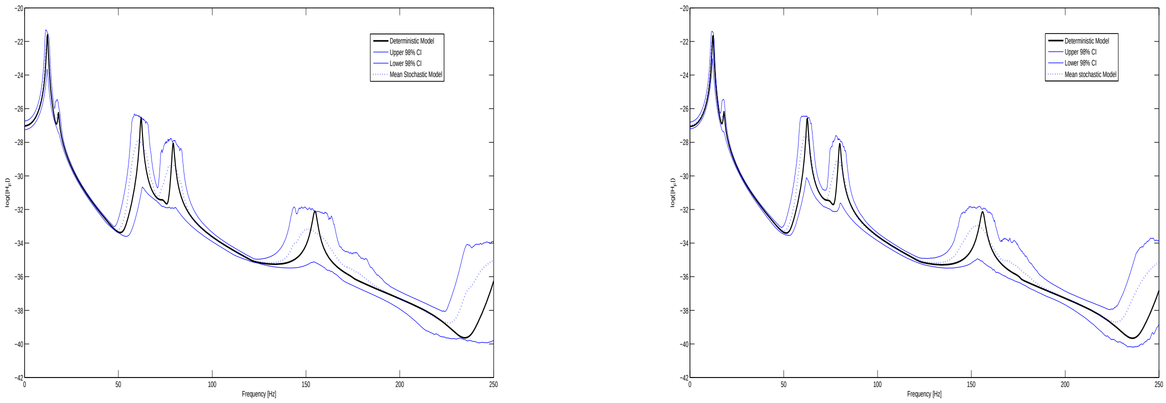


Figure 4: Convergence of the Monte Carlo simulations for CAS(45). (a) PPA with $\Delta_\alpha = 2$ and $\delta_i = 0.1$, (b) NPPA with $\delta_{\mathbb{K}} = 0.8$.

In Fig. 5 one can see a comparison of the simulation carried out for a curved composite thin walled beam with CAS(15). In simulation with the PPA the dispersion coefficient was $\delta_i = 0.1$ for all the elastic properties assumed uncertain, whereas the bounds of dispersion in the angle reinforcement were $\pm 2^0$. In the simulation with the NPPA, a dispersion parameter $\delta_{\mathbb{K}} = 0.50$ has been used. Each figure shows the deterministic response, the mean of the stochastic response and the upper and lower 98% confidence interval. For this particular case there are no relevant differences between both approaches that can be observed. In Fig. 6 one can see the FRF of the most representative kinematic variables of the CAS(15) configuration.

As it is possible to see the elastic coupling is present in every mode.



(a) (b)

Figure 5: FRF's Comparison of simulations for CAS(15). (a) PPA with $\Delta_\alpha = 2$ and $\delta_i = 0.1$, (b) NPPA with $\delta_{\mathbb{K}} = 0.50$.

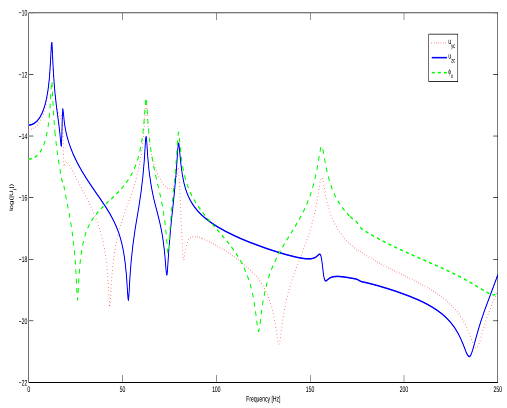
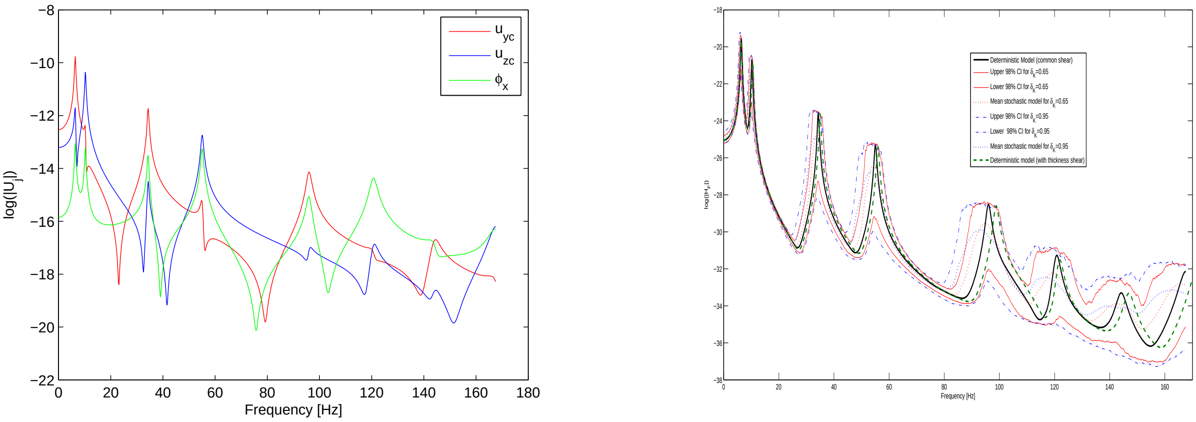


Figure 6: FRF's for the CAS(15) stacking sequence

Fig. 7(a) shows the FRFs of the kinematic variables in the cases of a CAS(45) stacking sequence. Fig 7(b) shows the FRFs of the point where the load is acting. In this figure one can see the comparison of the confidence intervals and the stochastic mean for dispersion parameters $\delta_{\mathbb{K}} = 0.65$ and $\delta_{\mathbb{K}} = 0.95$ in the NPPA simulation procedure. Also it is depicted the deterministic response when one employs a variation in the constitutive relationships by including or neglecting the shear deformability due to shell thickness. As it is possible to see the uncertainty in the model related to both constitutive approaches of shear can be captured with the NPPA.

In Fig. 8 one can see the FRF in the point where the load is acting but in this case, the NPPA is employing tanking into account uncertainty in the stiffness matrix as well as in the mass matrix. This could be the case of the hygroscopic effect and its influence in the incorporation of mass in the composite structure. Although there are constitutive alternatives (Jones, 1999; Piovan and Cortínez, 2007a) to tackle this problem, it assumes idealized behavior of the material. Thus, Fig. 8 was constructed with dispersion parameters $\delta_{\mathbb{M}} = \delta_{\mathbb{K}} = 0.65$. Notice that the influence of mass uncertainty can be sensitive in the upper modes.



(a) (b)

Figure 7: FRF's for CAS(45). (a) Kinematic variables: u_{yc}, u_{zc} and ϕ_x (b) Comparison of many alternatives with NPPA.

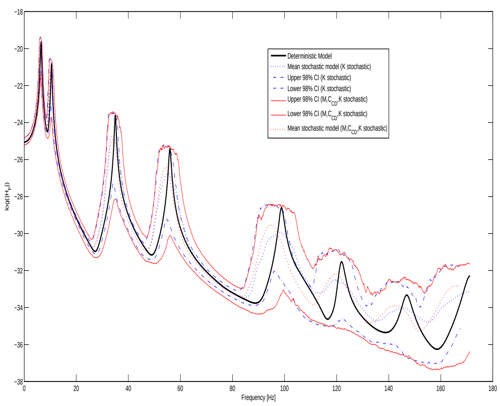


Figure 8: FRF's for the CAS(45) with uncertanty in the stiffness matrix and mass matrix

5 CONCLUSIONS

A preliminary study about the quantification of uncertainty and its propagation in the transient dynamics of curved thin walled composite beams has been carried out. Two possible approaches have been evaluated: the parametric probabilistic approach and the non-parametric probabilistic approach. The last one allows the analysis of uncertainties in the model as a whole.

Despite the differences between both approaches the following points should be remarked:

- The propagation of uncertainty in the transient dynamic response of composite curved beams is strongly influenced by the elastic coupling inherent to the lamination scheme: the more elastic coupling (e.g. CAS laminiation) the larger uncertainty propagation.
- The predictability decreases as the frequency increases for both approaches.
- The non-parametric probabilistic approach can face uncertainties in the modeling theories for example the type of shear flexibility theory employed or the effect of mass added by hygroscopic effects, for example.
- The dispersion parameter $\delta_{\mathbb{K}}$ of the random matrices should be determined with other tools, for example with experimental methodologies.

As final details, composite structures have notable features of uncertainty as observed in the previous sections. The PPA and NPAA involved in the analytic topics of this work have been useful tools to quantify the uncertainty and to explore its propagation in the transient dynamics of the curved thin-walled composite beams. Nevertheless, there are other concerns associated with the uncertainty of the model and the parameters themselves that was not analyzed, for example, the elastic properties may be correlated random variables and their influence should be quantified. However these matters would be part of further extensions to the present article.

ACKNOWLEDGEMENTS

The support of Universidad Tecnológica Nacional and CONICET of Argentina and FAPERJ and CNPq of Brasil is recognized.

REFERENCES

Barbero E. *Introduction to Composite Material Design*. Taylor and Francis Inc., NY. U.S.A., 1999.

Bathe K.J. *Finite Element procedures in Engineerign Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1996.

Bauchau O. A beam theory for anisotropic materials. *Journal of Applied Mechanics*, 52(2):416–422, 1985.

Bauld N. and Tzeng L. A vlasov theory for fiber-reinforced beams with thin-walled open cross sections. *International Journal of Solids and Structures*, 20(3):277–297, 1984.

Cesnik C., Sutyrin V., and Hodges D. Refined theory of composite beams: The role of short-wavelength extrapolation. *International Journal of Solids and Structures*, 33(10):1387–1408, 1996.

Cortínez V. and Piovan M. Vibration and buckling of composite thin-walled beams with shear deformability. *Journal of Sound and Vibration*, 258(4):701–723, 2002.

Jaynes E. Information theory and statistical mechanics. *The Physical Review*, 1957a.

Jaynes E. Information theory and statistical mechanics ii. *The Physical Review*, 1957b.

Jones R. *Mechanics of Composite Material*. Taylor and Francis Inc., NY. U.S.A., 1999.

Kim C. and White S. Thick walled composite beam theory including 3d elastic effects and torsional warping. *International Journal of Solids and Structures*, 34(31), 1997.

Librescu L. and Song O. On the aeroelastic tailoring of composite aircraft swept wings modeled as thin walled beam structures. *Composites Engineering*, 2(5), 1992.

Mehrez L., Doostan A., Moens D., and Vandepitte D. Stochastic identification of composite material properties from limited experimental databases, partii: Uncertainty modelling. *Mechanical Systems and Signal Processing*, 27:484–498, 2012a.

Mehrez L., Moens D., and Vandepitte D. Stochastic identification of composite material properties from limited experimental databases, parti: Experimental database construction. *Mechanical Systems and Signal Processing*, 27:471–483, 2012b.

Pawar P. On the behavior of thin walled composite beams with stochastic properties under matrix cracking damage. *Thin-Walled Structures*, 49:1123–1131, 2011.

Piovan M. *Estudio Teórico y Computacional sobre la mecánica de vigas curvas de materiales compuestos con secciones de paredes delgadas considerando efectos no convencionales*. PhD Thesis. Departamento de Ingeniería, Universidad Nacional del Sur, 2003.

Piovan M. and Cortínez V. Mechanics of shear deformable thin-walled beams made of composite materials. *Thin Walled Structures*, 45:37–62, 2007a.

Piovan M. and Cortínez V. Mechanics of thin-walled curved beams made of composite materials, allowing for shear deformability. *Thin Walled Structures*, 45:759–789, 2007b.

Piovan M., Ramirez J., and Sampaio R. Dynamics of thin-walled composite beams: Analysis of parametric uncertainties. *Composite Structures*, 105:14–28, 2013.

Rehfield L., Atilgan A., and Hodges D. Non-classical behavior of thin-walled composite beams with closed cross sections. *Journal American Helicopter Society*, 35(3):42–50, 1990.

Ritto T., Sampaio R., and Cataldo E. Timoshenko beam with uncertainty on the boundary conditions. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 30(4):295–303, 2008.

Salim S., Yadav D., and Iyengar N. Analysis of composite plates with random material characteristics. *Mechanics Research Communications*, 20(3):405–414, 1993.

Sampaio R. and Cataldo E. Comparing two strategies to model uncertainties in structural dynamics. *Shock and Vibration*, 17(2):171–186, 2011.

Shannon C. A mathematical theory of communication. *Bell Systems Tech*, 27:379–423, 1948.

Soize C. Maximum entropy approach for modeling random uncertainties in transient elastodynamics. *Journal of the Acoustical Society of America*, 109(5):1979–1996, 2001.

Soize C. Random matrix theory and non-parametric model of random uncertainties in vibration analysis. *Journal of Sound and Vibration*, 263:893–916, 2003.

Soize C. A comprehensive overview of a non-parametric probabilistic approach of model uncertainties for predictive models in structural dynamics. *Journal of Sound and Vibration*, 288(3):623–652, 2005.

Song O. and Librescu L. Free vibration of anisotropic composite thin-walled beams of closed cross-section contour. *Journal of Sound and Vibration*, 167(1):129–147, 1993.

Sriramula S. and Chryssantopoulos M. Qauntification of uncertainty modelling in stochastic analysis of frp composites. *Composites: Part A*, 40:1673–1684, 2009.

Vickenroy G. and Wilde W. The use of the monte carlo techniques in statistical finite element methods for the determination of the structural behavior of composite material structural components. *Composite Structures*, 32(1), 1995.

APPENDIX A

The vector of external forces $\tilde{\mathbf{P}}_X$ and the matrix of mass coefficients $\mathbf{M}_m$ can be calculated in the following form:

$$\tilde{\mathbf{P}}_X = \int_A \left[\begin{array}{ccc} \bar{X}_x & \bar{X}_y & \bar{X}_z \end{array} \right] \mathbf{G}_m \frac{dydz}{\mathcal{F}}, \tag{36}$$

$$\mathbf{M}_m = \int_A \rho\,(y,z)\, \mathbf{G}_m^T \mathbf{G}_m \frac{dydz}{\mathcal{F}}, \tag{37}$$

where $\bar{X}_x$, $\bar{X}_y$ and $\bar{X}_z$ are general volume forces, whereas:

$$\mathbf{G}_m = \left[\begin{array}{ccccccc} 1+y/R & 0 & -y & 0 & z-\omega/R & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 1 & 0 & y & 0 \end{array} \right], \tag{38}$$

The vector of natural boundary conditions $\tilde{\mathbf{B}}_X$ can be written in the subsequent form:

$$\tilde{\mathbf{B}}_X = \left\{ \begin{array}{c} -\bar{Q}_x + \bar{M}_z/R + Q_x - M_z/R \\ -\bar{Q}_y + Q_y \\ -\bar{M}_z + M_z \\ -\bar{Q}_z + Q_z \\ -\bar{M}_y + \bar{B}/R + M_y - B/R \\ -\bar{T}_{sv} - \bar{T}_w + T_{sv} + T_w \\ -\bar{B} + B \end{array} \right\}, \tag{39}$$

where $\bar{Q}_x$, $\bar{Q}_y$, $\bar{Q}_z$, $\bar{M}_y$, $\bar{M}_z$, $\bar{T}_w$ and $\bar{T}_{sv}$ are prescribed forces and moments applied at the boundaries.