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FINITE ELEMENT COMPUTATION OF BELTRAMI FIELDS

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Abstract. Vector fields ***H*** satisfying **curl*H*** = λ***H***, with λ being a scalar field, are called *force-free fields*. This name arises from magnetohydrodynamics, since a magnetic field of this kind induces a vanishing Lorentz force: ***F*** := ***J*** x ***B*** = **curl*H*** x (μ***H***). In 1958 Woltjer [6] showed that the lowest state of magnetic energy density within a closed system is attained when λ is spatially constant. In such a case H is called a *linear* force-free field and its determination is naturally related with the spectral problem for the curl operator. The eigenfunctions of this problem are known as *free-decay* fields and play an important role, for instance, in the study of turbulence in plasma physics.

The spectral problem for the curl operator, **curl*H*** = λ***H***, has a longstanding tradition in mathematical physics. A large measure of the credit goes to Beltrami [1], who seems to be the first who considered this problem in the context of uid dynamics and electromagnetism. This is the reason why the corresponding eigenfunctions are also called *Beltrami fields*. On bounded domains, the most natural boundary condition for this problem is ***H***.η = 0, which corresponds to a field confined within the domain. Analytical solutions of this problem are only known under particular symmetry assumptions. The first one was obtained in 1957 by Chandrasekhar and Kendall [2] in the context of astrophysical plasmas arising in modeling of the solar crown.

A couple of numerical methods based on Nédélec finite elements have been introduced and analyzed in a recent paper [5] for the solution of the eigenvalue problem for the curl operator in simply connected domains. This topological assumption is not just a technicality, since the eigenvalue problem is ill-posed on multiply connected domains, in the sense that its spectrum is the whole complex plane, as is shown in [3]. However, additional constraints can be added to the eigenvalue problem in order to recover a well posed problem with a discrete spectrum [3, 4]. We choose as additional constraints a zero-ux condition of the curl on all the cutting surfaces. We introduce two weak formulations of the corresponding problem, which are convenient variations of those studied in [5]; one of them is mixed and the other a Maxwell-like formulation. We prove that both are well posed and show how to modify the finite element discretization from [5] to take care of these additional constraints. We prove spectral convergence of both discretization as well as a priori error estimates. Finally, we report a numerical test which allows us to assess the performance of the proposed methods.

References

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