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MODELING OF NON-LOCAL BEAM THEORIES FOR VIBRATORY AND BUCKLING PROBLEMS OF NANO-TUBES

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Abstract. This article is concerned with the study of vibratory and buckling problems associated to micro and nano-beams modeled with first-order shear approaches in the context of non-local elasticity. Normally this kind of studies is performed in the context of a single Timosheko beam model or even coupled with axial motions and also with extended Bernoulli-Euler beam models, among others. In the present study the slender structure is conceived as a thin-walled beam with motion in three axes, i.e. bending in two directions and the axial motion and twisting. Thus a model for both isotropic and functionally graded beams is developed. Then the model is employed to calculate buckling loads and vibration frequencies with or without the presence of initial stresses. The derivation procedure of the non-local beam model follows three steps. First: the differential equations of a local model are derived in terms of internal forces; second: the internal forces are re-deduced in terms of the non-local constitutive equations which are substituted, as a third step, in the previous differential equations giving the differential equations of the non-local beam model. The power series method is employed to offer an analytical response for basic buckling and vibration problems, as well as the finite element method in used to calculate the vibratory and buckling features for more complex configurations. A number of examples are presented in order to show the influence of the non-local formulation in the buckling response and the vibratory patters of nano/microbeams.

1 INTRODUCTION

The introduction of the concept of nano-structures and the possibility of constructing and tailoring them to fulfill given services brought the inspiration of a revolution in high-tech research. Since the discovery of the so called carbon nano-tubes (Iijima, 1991) a lot of different devices have been envisioned and consequently an enormous effort is done to analyzed them because of their high potential in high technological appliances. Carbon nano-tubes, nano-beams, nano-composites, MEE nano beam, among others (Murmu et al., 2011; Thai, 2012; Hemmatnezhad and Ansari, 2013) are just a few examples of the nano-sized structures that possess mechanical, chemical and electrical properties among others, that make them a good choice for sensing/actuating at quite small scales. The structures in the nano-scale have intramolecular forces as well as a spatial size distribution of the molecular microstructure (Di Paola et al., 2011; Peddison et al., 2003) that cannot be successfully tackled with classical continuum models (Murmu et al., 2011; Murmu and Adhikari, 2012). In order to study the dynamics of the nano structures, the so-called molecular dynamics approach (Murmu and Adhikari, 2012) has been employed with satisfactory results. Although molecular dynamics approach is good to characterized the behavior of nano structures, its high computational cost makes it quite prohibitive for computationally demanding problems or for real-time vibratory sensing devices.

The non-local theory of an elastic continuum introduced by Eringen (1972) as an approach for evaluating the mechanics of cracks, can be a nice tool to overcome the problems of the prominent size effects of the nano-scale that conventional continuum theories cannot handle as shown by Peddison et al. (2003). Thus, within the context of non-local elasticity according to Eringen it is proposed that the size-effects can be captured by assuming the stresses at a given point as functions of the strains of the whole domain (Eringen, 1972; Reddy, 2007, 2010). Moreover, the non-local theory of Eringen contains, as a limit case, the conventional local elasticity employed in classical structural mechanics (Peddison et al., 2003; Reddy, 2007).

During the past 10 years many authors have been developing models for nano-beams, nano-tubes, etc. for uses as bio-sensors or resonators with the aim to detect anomalies in the vibration pattern connected with the presence of attached masses (Elishakoff and Pentaras, 2009; Kang et al., 2009; Joshi et al., 2010; De Rosa and Lippiello, 2014; Adhikari and Chowdhury, 2010) and many other scopes (Lu et al., 2007; Di Paola et al., 2011; Elishakoff et al., 2011; Peddison et al., 2003). In some of those contributions the non-local elasticity formulation of Eringen has been employed. In fact, Peddison et al. (2003) developed a simple non-local Bernoulli-Euler type beam model analyzing the dynamics of a nano-beam with emphasis in evaluating the influence of the non-local parameters. Reddy (2007), Reddy (2010), Lu et al. (2007), Wang et al. (2007) and Simsek (2013b) among others developed different studies for non-local Timoshenko beams.

Although in the aforementioned papers comprehensive studies of free vibrations, elastic and dynamic stability among other have been covered; it has to be said that only simple models for bending in one plane or twisting or extension have been employed. Taking into account this context, the aim of the present research is focused in the development of a non-local beam model for coupled extension-twisting-bending motions. The model takes into account bending shear flexibility and a state of initial stresses. The nano-beam model for non-local elasticity is derived from the the one based on classical elasticity by incorporating the non-local constitutive equations. This is done in the differential equations of motion. At this point several approaches (for example power series of the whole element method) can be applied to solve the equations in the frame of dynamics or static stability. On the other hand, a weak form should be deduced in order to develop the finite element approach in the context of non-local elasticity.

Power Series and finite element methods are employed to calculate buckling loads and vibratory frequencies with or without the presence of initial stresses. The effect of non-locality is evaluated through the variation of the non-local parameter and its effect in single motions and in coupled motions due to non-local elastic constitutive relations.

2 MODEL DEVELOPMENT

2.1 Kinematics

The nano-beam consists of a thin walled tube as the one shown in Figure 1. In order to construct the deterministic model for beam the following assumptions are proposed: (a) The material may be isotropic or functionally graded along the wall thickness; (b) The displacements and strains are considered small according to the linear elasticity; (c) The constitutive relations correspond to the Eringen's non-local formulation; (d) The bending shear deformability is considered and the non-uniform warping is neglected; (e) a state of normal initial stresses is considered.

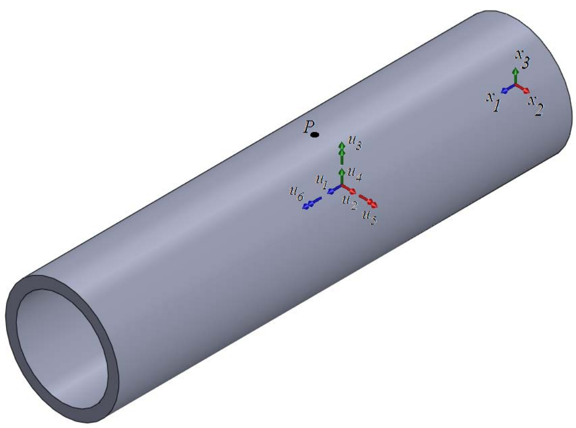

Figure 1: Single walled nano tube

Under the aforementioned conditions the displacement field of the micro/nano-beam can be represented in the following form:

$$\bar{U}_{\boldsymbol{P}} = \left\{ \begin{array}{c} u_{x_1} \\ u_{x_2} \\ u_{x_3} \end{array} \right\} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_4 \end{array} \right\} + \left[\begin{array}{ccc} 0 & -u_3 & u_5 \\ u_3 & 0 & -u_6 \\ -u_5 & u_6 & 0 \end{array} \right] \left\{ \begin{array}{c} 0 \\ x_2 \\ x_3 \end{array} \right\} \tag{1}$$

where: u_{x_1}, u_{x_2} and u_{x_3} are the displacement coordinates of a generic point $\boldsymbol{P}$ in the nano-beam's domain, whereas u_1 is the axial displacement of the reference system u_2, and u_4 are the lateral displacements, u_6 is the twisting angle and u_3 and u_5 are the bending rotation parameters.

According to the aforementioned assumptions the most representative strain components are ε_{11}, ε_{12} and ε_{13} that can be taken from the definition of the Green-Lagrange strain tensor as:

$$\begin{array}{l} \varepsilon_{11} = \varepsilon_{11}^{L} + \varepsilon_{11}^{NL} = u_{1,1} + \frac{1}{2} u_{i,1} u_{i,1} \\ \varepsilon_{12} = \gamma_{12}^{L}/2 = \left(u_{1,2} + u_{2,1} \right)/2 \\ \varepsilon_{13} = \gamma_{13}^{L}/2 = \left(u_{1,3} + u_{3,1} \right)/2 \end{array} \tag{2}$$

where:

$$\begin{aligned}
&\left\{\varepsilon_{11}^{L}\right\} = \left[\begin{array}{ccc} 1 & x_3 & -x_2 \end{array}\right]\left[\begin{array}{ccc} \varepsilon_{D1} & \varepsilon_{D5} & \varepsilon_{D3} \end{array}\right]^T \\
&\left\{\begin{array}{c} \gamma_{12}^{L} \\ \gamma_{13}^{L} \end{array}\right\} = \left[\begin{array}{ccc} 1 & 0 & -x_3 \\ 0 & 1 & x_2 \end{array}\right]\left\{\begin{array}{c} \varepsilon_{D2} \\ \varepsilon_{D4} \\ \varepsilon_{D6} \end{array}\right\} \\
&\varepsilon_{11}^{NL} = \frac{1}{2}\left[\left(\varepsilon_{11}^{L}\right)^2 + \left(u_2' - x_3 u_6'\right)^2 + \left(u_4' + x_2 u_6'\right)^2\right]
\end{aligned} \tag{3}$$

In which, for purposes of simplification in the subsequent paragraphs, the following definitions are made:

$$\{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}, \varepsilon_{D5}, \varepsilon_{D6}\} = \{u_1', u_2' - u_3, u_3', u_4' + u_5, u_5', u_6'\} \tag{4}$$

Henceforth apostrophes are used to identify the differentiation with respect to spatial coordinate x_1. It is clear that the rest of the strain components are considered negligible.

2.2 Non Local constitutive relations

According to the nonlocal elasticity theory, the stress at a given point depends on the strains of the whole continuum (Eringen, 1972). This assumption may be written as:

$$\bar{\sigma} - \mu \nabla^2 \bar{\sigma} = \bar{\mathcal{C}} : \bar{\varepsilon} \tag{5}$$

where $\bar{\sigma}$ is the stress tensor, $\bar{\mathcal{C}}$ is the fourth order Hookean elasticity tensor, and $\bar{\varepsilon}$ is the strain tensor. The symbols ∇^2 and $:$ are the Laplacian operator and double dot tensor product. The parameter $\mu = (e_o a)^2$ is a scale factor that depends on the material and geometric features. The coefficient e_o is estimated such that the non local elasticity matches the atomistic lattice models, and a is the so called internal characteristic lengths (Eringen, 1972; Simsek, 2013a).

In the case of slender nano-beams taking into account the assumption of the previous paragraph, the Eq. (7) can be reduced to the following form (Simsek, 2013b; Ke et al., 2012; Peddison et al., 2003):

$$\begin{aligned}
\sigma_{11} - \mu \sigma_{11}'' &= E_{11} \varepsilon_{11}^{L} \\
\sigma_{12} - \mu \sigma_{12}'' &= G_{12} \gamma_{12}^{L} \\
\sigma_{13} - \mu \sigma_{13}'' &= G_{13} \gamma_{13}^{L}
\end{aligned} \tag{6}$$

In Eq. (6), E_{11} and $G_{12} = G_{13} = G$ are the longitudinal and transversal elasticity moduli, whereas ε_{11}^{L}, γ_{12}^{L} and γ_{13}^{L} are the strain components given in Eq. (3).

2.3 Motion Equations

The motion equations of the beam can be deduced by means of the linearized principle of virtual works:

$$\int_V \bar{\sigma} : \delta\bar{\varepsilon} dV + \int_V \bar{\sigma}^0 : \delta\bar{\eta} dV + \int_V \rho \ddot{\bar{\mathcal{U}}} \cdot \delta\bar{\mathcal{U}} dV - \int_V \bar{X} \cdot \delta\bar{\mathcal{U}} dV = 0 \tag{7}$$

$$\int_V \bar{\sigma}^0 : \delta\bar{\varepsilon} dV - \int_V \bar{X}^0 \cdot \delta\bar{\mathcal{U}} dV = 0 \tag{8}$$

The Eq. (7) is the incremental part of the Principle of Virtual Work that is subjected to the constraint Eq. (8) which implies the condition of self-equilibrium of initial stresses and initial forces $\bar{X}^0$. In Eq. (7), the first and second integral correspond to the virtual works of the internal forces and initial stresses, respectively; the third integral corresponds to the virtual work of the inertial forces and the fourth integral corresponds to the virtual work of the applied forces volume $\bar{X}$. The over dots identify derivation with respect to the time.

Applying the conventional steps of variational calculus, the motion equations can be written as follows:

$$\begin{aligned}
&-Q_1' + \mathcal{G}_1 + \mathcal{M}_1 - \mathcal{F}_1 = 0\\
&-Q_2' + \mathcal{G}_2 + \mathcal{M}_2 - \mathcal{F}_2 = 0\\
&-M_3' - Q_2 + \mathcal{G}_3 + \mathcal{M}_3 - \mathcal{F}_3 = 0\\
&-Q_3' + \mathcal{G}_4 + \mathcal{M}_4 - \mathcal{F}_4 = 0\\
&-M_2' + Q_3 + \mathcal{G}_5 + \mathcal{M}_5 - \mathcal{F}_5 = 0\\
&-M_1' + \mathcal{G}_6 + \mathcal{M}_6 - \mathcal{F}_6 = 0
\end{aligned} \tag{9}$$

These equations are subjected to conventional boundary conditions at $x_1 = 0$ and $x_1 = L$ by prescribing the appropriate kinematic variables or forces/moments depending on the case.

In Eq. (9), Q_1 is the axial force, M_2 and M_3 are the bending moments, Q_2 and Q_3 are the shear forces and M_1 is the twisting moment, whereas $\mathcal{G}_i = -\mathcal{D}_i'$, $\mathcal{M}_i$ and $\mathcal{F}_i$ with $i = 1, ..., 6$ correspond to the terms of initial forces (employed for solving static instability or linearized buckling problems), inertial forces and applied forces respectively. For the sake of simplified legibility, the extended expressions of $\mathcal{D}_i$, $\mathcal{M}_i$ and $\mathcal{F}_i$ with $i = 1, ..., 6$ are given defined as follows:

$$\begin{aligned}
\mathcal{D}_1 &= Q_1^0 u_1' + M_3^0 u_3' + M_2^0 u_5',\\
\mathcal{D}_2 &= Q_1^0 u_2' - M_2^0 u_6',\\
\mathcal{D}_3 &= N_{33}^0 u_3' - M_3^0 u_1',\\
\mathcal{D}_4 &= Q_1^0 u_4' + M_3^0 u_6',\\
\mathcal{D}_5 &= N_{22}^0 u_3' + M_2^0 u_1',\\
\mathcal{D}_6 &= N_{11}^0 u_6' - M_3^0 u_4' - M_2^0 u_2',
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathcal{M}_1 = \rho_{11}\ddot{u}_1, \quad \mathcal{M}_2 = \rho_{22}\ddot{u}_2, \quad \mathcal{M}_4 = \rho_{44}\ddot{u}_4,\\
\mathcal{M}_3 = \rho_{33}\ddot{u}_3, \quad \mathcal{M}_5 = \rho_{55}\ddot{u}_5, \quad \mathcal{M}_6 = \rho_{66}\ddot{u}_6,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\{\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5\} &= \int_A \bar{X}_1 \{1, -x_2, x_3\}\, dx_2 dx_3\\
\{\mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_6\} &= \int_A \left\{\bar{X}_2, \bar{X}_3, -x_3\bar{X}_2 + x_2\bar{X}_3\right\} dx_2 dx_3
\end{aligned} \tag{12}$$

In the previous expression, Q_1^0, M_2^0 and M_3^0, as well as N_{11}^0, N_{22}^0 and N_{33}^0 identify the initial internal forces/moments under a uniform state of normal initial stresses. These forces/moments can be parameterized in order to calculate buckling loads, that is for example $Q_1^0 = \zeta F^0$ with $||F^0|| = 1$, and ζ as the parameter for the eigenvalue problem of static instability or buckling.

The stiffness coefficients J_{ih} and inertia coefficients ρ_{ih} are defined as:

$$\begin{aligned}
\rho_{11} = \rho_{22} = \rho_{44} &= \int_A \rho\, dx_2 dx_3\\
\{\rho_{33}, \rho_{55}, \rho_{66}\} &= \int_A \rho \left\{x_2^2, x_3^2, x_2^2 + x_3^2\right\} dx_2 dx_3
\end{aligned} \tag{13}$$

where κ is the shear factor for a single walled nano tube.

The internal forces are defined as stress resultants in the following form:

$$\begin{aligned}\{Q_1, M_2, M_3\} &= \int_A \sigma_{11}\,\{1, x_3, -x_2\}\,dx_2dx_3 \\ \{Q_2, Q_3, M_1\} &= \int_A \{\sigma_{12}, \sigma_{13}, -x_3\sigma_{12} + x_2\sigma_{13}\}\,dx_2dx_3\end{aligned} \tag{14}$$

where: A is the domain of the cross-section of the beam. On the other side Q_1^0, M_2^0, M_3^0, N_{11}^0, N_{22}^0 and N_{33}^0 are defined in terms of σ_{11}^0, that is:

$$\begin{aligned}\left\{Q_1^0, M_2^0, M_3^0\right\} &= \int_A \sigma_{11}^0\,\{1, x_3, -x_2\}\,dx_2dx_3 \\ \left\{N_{11}^0, N_{22}^0, N_{33}^0\right\} &= \int_A \sigma_{11}^0\,\left\{x_2^2 + x_3^2, x_3^2, x_2^2\right\}\,dx_2dx_3\end{aligned} \tag{15}$$

Using Eq. (14) and Eq. (6), it is possible to derive the non-local constitutive form in terms of beam forces/moments, that can be written as:

$$\begin{aligned}&Q_1 - \mu Q_1'' = J_{11}\varepsilon_{D1}, \quad Q_2 - \mu Q_2'' = J_{22}\varepsilon_{D2}, \quad Q_3 - \mu Q_3'' = J_{44}\varepsilon_{D4}, \\ &M_3 - \mu M_3'' = J_{33}\varepsilon_{D3}, \quad M_2 - \mu M_2'' = J_{55}\varepsilon_{D5}, \quad M_1 - \mu M_1'' = J_{66}\varepsilon_{D6},\end{aligned} \tag{16}$$

The definition of the stiffness coefficients J_{ik}, $i, k = 1, ..., 6$ can be followed in the Appendix I. Now, with the aid of Eq. (16) and the differential equations of motion (9) it is possible (Reddy, 2007, 2010; Simsek, 2013b) to derive the non-local form of the beam forces and moments as:

$$\begin{aligned}Q_1 &= J_{11}\varepsilon_{D1} + \mu\mathcal{N}_1', \\ Q_2 &= J_{22}\varepsilon_{D2} + \mu\mathcal{N}_2', \\ M_3 &= J_{33}\varepsilon_{D3} + \mu\left(\mathcal{N}_3' - \mathcal{N}_2\right), \\ Q_3 &= J_{44}\varepsilon_{D4} + \mu\mathcal{N}_4', \\ M_2 &= J_{55}\varepsilon_{D5} + \mu\left(\mathcal{N}_5' - \mathcal{N}_4\right), \\ M_1 &= J_{66}\varepsilon_{D6} + \mu\mathcal{N}_6',\end{aligned} \tag{17}$$

where, the form $\mathcal{N}_i = \mathcal{G}_i + \mathcal{M}_i - \mathcal{F}_i$, $i = i, ..., 6$, is defined in order to compress notation.

Finally the non-local governing equations can be written in the following form:

$$\begin{aligned}&-J_{11}\varepsilon_{D1}' - \mu\mathcal{N}_1'' + \mathcal{N}_1 = 0, \\ &-J_{22}\varepsilon_{D2}' - \mu\mathcal{N}_2'' + \mathcal{N}_2 = 0, \\ &-J_{33}\varepsilon_{D3}' - J_{22}\varepsilon_{D2} - \mu\mathcal{N}_3'' + \mathcal{N}_3 = 0, \\ &-J_{44}\varepsilon_{D4}' - \mu\mathcal{N}_4'' + \mathcal{N}_4 = 0, \\ &-J_{55}\varepsilon_{D5}' + J_{44}\varepsilon_{D4} - \mu\mathcal{N}_5'' + \mathcal{N}_5 = 0, \\ &-J_{66}\varepsilon_{D6}' - \mu\mathcal{N}_6'' + \mathcal{N}_6 = 0,\end{aligned} \tag{18}$$

Note that when the coefficient $\mu = 0$, the previous non-local expressions of governing equations and the beam forces and moments, the corresponding local counterparts can be retrieved. The governing equations of the present model contains many of the recent approaches for buckling and vibration of nano-beams for Timoshenko theories in one plane, as one can find in the works of Reddy (2007), Reddy (2010), Phadikar (2010), Di Paola et al. (2011), and many others.

2.4 Non dimensional forms of the weak and strong forms of the motion equations

The weak variational formulation of the non-local beam can be retrieved by means of integration by parts of Eq. (18). Consequently the weak variational formulation has the following aspect:

$$
\begin{aligned}
&\int_L \left\{ \sum_{i=1}^{6} \left[J_{ii} \varepsilon_{D_i} \delta\varepsilon_{D_i} + \rho_{ii} \left(\ddot{u}_i \delta u_i + \mu \ddot{u}_i' \delta u_i' \right) \right] \right\} dx_1 + \\
&\int_L \left\{ \sum_{i=1}^{6} \left[\mathcal{D}_i \delta u_i' - \mu \mathcal{G}_i \delta u_i'' - \mathcal{F}_i \left(\delta u_i - \mu \delta u_i'' \right) \right] \right\} dx_1 + \\
&\left[- \left(Q_1 + \mathcal{D}_1 \right) \delta u_1 - \left(Q_2 + \mathcal{D}_2 \right) \delta u_2 - \left(M_3 + \mathcal{D}_3 + \mu \mathcal{N}_2 \right) \delta u_3 \right]\Big|_{x_1 = 0, L} + \\
&\left[- \left(Q_3 + \mathcal{D}_4 \right) \delta u_4 - \left(M_2 + \mathcal{D}_5 - \mu \mathcal{N}_4 \right) \delta u_5 - \left(M_1 + \mathcal{D}_6 \right) \delta u_6 \right]\Big|_{x_1 = 0, L} + \\
&\sum_{i=1}^{6} \left[\mu \left(\mathcal{G}_i - \mathcal{F}_i \right) \delta u_i' \right]\Big|_{x_1 = 0, L} = 0
\end{aligned}
\tag{19}
$$

Taking into account that the solution of the Eq. (18) or the Eq. (19) at least the approximation of the solution, employing Eq. (19), may follow different alternatives, and having in mind that the non-local nature of the nano-beam model together with their size effect lead to potential instabilities in the calculation methodologies or to stiff systems in several numerical approaches (Hemmatnezhad and Ansari, 2013), it is decided to develop a non-dimensional form of the weak formulation corresponding to Eq. (19). Then as a first step the kinematical variables are transformed in a non-dimensional form as:

$$
\begin{aligned}
\left\{ \bar{u}_1, \bar{u}_2, \bar{u}_4 \right\} &= \left\{ u_1, u_2, u_4 \right\} / L \\
\left\{ \bar{u}_3, \bar{u}_5, \bar{u}_6 \right\} &= \left\{ u_3, u_5, u_6 \right\}
\end{aligned}
\tag{20}
$$

The space-variable is also transformed in non-dimensional form as $\bar{x}_1 = x_1/L$. Consequently, the generalized deformations, the initial internal forces/moments and other terms involved in the virtual work of initial stresses can be redefined non-dimensionally as:

$$
\begin{aligned}
&\left\{ \bar{\varepsilon}_{D1}, \bar{\varepsilon}_{D2}, \bar{\varepsilon}_{D3}, \bar{\varepsilon}_{D4}, \bar{\varepsilon}_{D5}, \bar{\varepsilon}_{D6} \right\} = \left\{ \bar{u}_1', \bar{u}_2' - \bar{u}_3, \bar{u}_3', \bar{u}_4' + \bar{u}_5, \bar{u}_5', \bar{u}_6' \right\} \\
&\left\{ \bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2, \bar{\mathcal{F}}_3, \bar{\mathcal{F}}_4, \bar{\mathcal{F}}_5, \bar{\mathcal{F}}_5 \right\} = \left\{ \mathcal{F}_1 L, \mathcal{F}_2 L, \mathcal{F}_3, \mathcal{F}_4 L, \mathcal{F}_5, \mathcal{F}_6 \right\} / J_{11} \\
&\left\{ \bar{Q}_1^0, \bar{M}_2^0, \bar{M}_3^0, \bar{N}_{11}^0, \bar{N}_{22}^0, \bar{N}_{33}^0 \right\} = \left\{ Q_1^0 L^2, M_2^0 L, M_3^0 L, N_{11}^0, N_{22}^0, N_{33}^0 \right\} / \left(J_{11} L^2 \right)
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\bar{\mathcal{D}}_1 &= \bar{Q}_1^0 \bar{u}_1' + \bar{M}_3^0 \bar{u}_3' + \bar{M}_2^0 \bar{u}_5', \\
\bar{\mathcal{D}}_2 &= \bar{Q}_1^0 \bar{u}_2' - \bar{M}_2^0 \bar{u}_6', \\
\bar{\mathcal{D}}_3 &= \bar{N}_{33}^0 \bar{u}_3' + \bar{M}_3^0 \bar{u}_1', \\
\bar{\mathcal{D}}_4 &= \bar{Q}_1^0 \bar{u}_4' - \bar{M}_3^0 \bar{u}_6', \\
\bar{\mathcal{D}}_5 &= \bar{N}_{22}^0 \bar{u}_5' + \bar{M}_2^0 \bar{u}_1', \\
\bar{\mathcal{D}}_6 &= \bar{N}_{11}^0 \bar{u}_6' - \bar{M}_2^0 \bar{u}_2' - \bar{M}_3^0 \bar{u}_4',
\end{aligned}
\tag{22}
$$

Moreover the following non-dimensional material constants are defined:

$$
\begin{aligned}
&\Gamma_1 = 1, \Gamma_2 = \frac{J_{22}}{J_{11}}, \Gamma_3 = \frac{J_{33}}{L^2 J_{11}}, \Gamma_4 = \frac{J_{44}}{J_{11}}, \Gamma_5 = \frac{J_{55}}{L^2 J_{11}}, \Gamma_6 = \frac{J_{66}}{L^2 J_{11}} \\
&\Delta_1 = \Delta_2 = \Delta_4 = 1, \Delta_3 = \frac{\rho_{33}}{L^2 \rho_{11}}, \Delta_5 = \frac{\rho_{55}}{L^2 \rho_{11}}, \Delta_6 = \frac{\rho_{66}}{L^2 \rho_{11}},
\end{aligned}
\tag{23}
$$

Then the non-dimensional form of the Eq. (19) can be written as:

$$\begin{aligned}
&\int_L \left\{ \sum_{i=1}^{6} \left[\Gamma_i \bar{\varepsilon}_{D_i} \delta\bar{\varepsilon}_{D_i} + \Delta_i \lambda \left(\ddot{\bar{u}}_i \delta\bar{u}_i + \bar{\mu} \ddot{\bar{u}}_i' \delta\bar{u}_i' \right) \right] \right\} d\bar{x}_1 + \\
&\int_L \left\{ \sum_{i=1}^{6} \left[\bar{\mathcal{D}}_i \delta\bar{u}_i' + \bar{\mu} \bar{\mathcal{G}}_i \delta\bar{u}_i'' - \bar{\mathcal{F}}_i \left(\delta\bar{u}_i - \bar{\mu} \delta\bar{u}_i'' \right) \right] \right\} d\bar{x}_1 + \\
&\left[- \left(\bar{Q}_1 + \bar{\mathcal{D}}_1 \right) \delta\bar{u}_1 - \left(\bar{Q}_2 + \bar{\mathcal{D}}_2 \right) \delta\bar{u}_2 - \left(\bar{M}_3 + \bar{\mathcal{D}}_3 + \bar{\mu} \bar{\mathcal{N}}_2 \right) \delta\bar{u}_3 \right] \Big|_{\bar{x}_1 = 0, L} + \\
&\left[- \left(\bar{Q}_3 + \bar{\mathcal{D}}_4 \right) \delta\bar{u}_4 - \left(\bar{M}_2 + \bar{\mathcal{D}}_5 - \bar{\mu} \bar{\mathcal{N}}_4 \right) \delta\bar{u}_5 - \left(\bar{M}_1 + \bar{\mathcal{D}}_6 \right) \delta\bar{u}_6 \right] \Big|_{\bar{x}_1 = 0, L} + \\
&\sum_{i=1}^{6} \left[\mu \left(-\bar{\mathcal{D}}_i' - \bar{\mathcal{F}}_i \right) \delta\bar{u}_i' \right] \Big|_{\bar{x}_1 = 0, L} = 0
\end{aligned} \tag{24}$$

where:

$$\bar{\mu} = \frac{\mu}{L^2} = \left(\frac{e_o a}{L} \right)^2, \lambda = \frac{\rho_{11} L^2}{J_{11}}, \tag{25}$$

The functional given in Eq. (24) can be employed to derive a finite element approximation to the solution of the motion equations for several problems of mechanics of nano-beams. On the other hand the non-dimensional differential equations can be represented as follows:

$$\begin{aligned}
&-\Gamma_1 \bar{u}_1'' + \lambda \Delta_1 \left(\ddot{\bar{u}}_1 - \bar{\mu} \ddot{\bar{u}}_1'' \right) - \bar{\mathcal{D}}_1' + \bar{\mu} \bar{\mathcal{G}}_1'' + \left(\bar{\mathcal{F}}_1 - \bar{\mu} \bar{\mathcal{F}}_1'' \right) = 0 \\
&-\Gamma_2 \left(\bar{u}_2'' - \bar{u}_3' \right) + \lambda \Delta_2 \left(\ddot{\bar{u}}_2 - \bar{\mu} \ddot{\bar{u}}_2'' \right) - \bar{\mathcal{D}}_2' + \bar{\mu} \bar{\mathcal{G}}_2'' + \left(\bar{\mathcal{F}}_2 - \bar{\mu} \bar{\mathcal{F}}_2'' \right) = 0 \\
&-\Gamma_3 \bar{u}_3'' - \Gamma_2 \left(\bar{u}_2' - \bar{u}_3 \right) + \lambda \Delta_3 \left(\ddot{\bar{u}}_3 - \bar{\mu} \ddot{\bar{u}}_3'' \right) - \bar{\mathcal{D}}_3' + \bar{\mu} \bar{\mathcal{G}}_3'' + \left(\bar{\mathcal{F}}_3 - \bar{\mu} \bar{\mathcal{F}}_3'' \right) = 0 \\
&-\Gamma_4 \left(\bar{u}_4'' + \bar{u}_5' \right) + \lambda \Delta_4 \left(\ddot{\bar{u}}_4 - \bar{\mu} \ddot{\bar{u}}_2'' \right) - \bar{\mathcal{D}}_4' + \bar{\mu} \bar{\mathcal{G}}_4'' + \left(\bar{\mathcal{F}}_4 - \bar{\mu} \bar{\mathcal{F}}_4'' \right) = 0 \\
&-\Gamma_5 \bar{u}_5'' + \Gamma_4 \left(\bar{u}_4' + \bar{u}_5 \right) + \lambda \Delta_5 \left(\ddot{\bar{u}}_5 - \bar{\mu} \ddot{\bar{u}}_5'' \right) - \bar{\mathcal{D}}_5' + \bar{\mu} \bar{\mathcal{G}}_5'' + \left(\bar{\mathcal{F}}_5 - \bar{\mu} \bar{\mathcal{F}}_5'' \right) = 0 \\
&-\Gamma_6 \bar{u}_6'' + \lambda \Delta_6 \left(\ddot{\bar{u}}_6 - \bar{\mu} \ddot{\bar{u}}_6'' \right) - \bar{\mathcal{D}}_6' + \bar{\mu} \bar{\mathcal{G}}_6'' + \left(\bar{\mathcal{F}}_6 - \bar{\mu} \mathcal{F}_6'' \right) = 0
\end{aligned} \tag{26}$$

which are subjected to the following boundary conditions:

$$\begin{aligned}
&\left(\bar{Q}_1 + \bar{\mathcal{D}}_1 \right) \delta\bar{u}_1 = 0 \\
&\left(\bar{Q}_2 + \bar{\mathcal{D}}_2 \right) \delta\bar{u}_2 = 0 \\
&\left(\bar{M}_3 + \bar{\mathcal{D}}_3 + \bar{\mu} \bar{\mathcal{N}}_2 \right) \delta\bar{u}_3 = 0 \\
&\left(\bar{Q}_3 + \bar{\mathcal{D}}_4 \right) \delta\bar{u}_4 = 0 \\
&\left(\bar{M}_2 + \bar{\mathcal{D}}_5 - \bar{\mu} \bar{\mathcal{N}}_4 \right) \delta\bar{u}_5 = 0 \\
&\left(\bar{M}_1 + \bar{\mathcal{D}}_6 \right) \delta\bar{u}_6 = 0 \\
&\bar{\mu} \left(-\bar{\mathcal{D}}_i' - \bar{\mathcal{F}}_i \right) \delta\bar{u}_i' = 0, i = 1, ..., 6
\end{aligned} \tag{27}$$

The Eq. (26) subjected to Eq. (27) can be solved by techniques such as the Power Series Method or differential quadrature among many others.

3 NUMERICAL METHODS FOR BUCKLING AND VIBRATION PROBLEMS

In the following sections two different approaches are developed to solve the motion equations of the non-local nano-beam, namely power series method and finite element method. For both methods, the calculation of vibration frequencies with or without the presence of initial stresses needs a harmonic motion, then the kinematic variables can be represented as:

$$\bar{u}_j \left(\bar{x}, t \right) = \bar{U}_j \left(\bar{x} \right) e^{I \Omega t}, j = 1, ..., 6 \tag{28}$$

where Ω is the circular frequency, $I = \sqrt{-1}$ and $\bar{U}_j$, $j = 1, ..., 6$ are the corresponding modal shapes. Now, substituting Eq.(28) in both Eq.(24) or Eq.(26) one can get the equations for calculating the problems of static or dynamic eigenvalues.

In the following paragraphs it is assumed also that no external distributed forces are employed in the motion equations derived form the incremental expression of the Principle of Virtual Work, or Eq. (7), i.e. $\bar{\mathcal{F}}_j$, $j = 1, ...6$ in Eq.(26) and/or Eq.(24).

3.1 Power series method

Assuming the existence and uniqueness of the solution of Eq. (26) subjected to Eq. (27) according to the harmonic motion of Eq. (28) that govern the problem of free vibration and buckling of the non-local straight beam in the space, it is proposed that the aforementioned solution of the 6 kinematic variables (mode shapes) can be represented in the form of power series. Thus, the variables are supposed to be described in the following form:

$$\bar{U}_j\left(\bar{x}\right) = \sum_{k=0}^{Z} \bar{U}_{jk}^{*} \bar{x}^k, j = 1, ..., 6 \tag{29}$$

Theoretically, $Z \to \infty$ gives the analytical solution, however for practical purposes Z may be a large integer, which implies the approximation of the response. Evidently, the approximations of the solution would be polynomials with an order such to reach a given accuracy in the eigenvalue (of both dynamic or static) calculation.

Since, the differential system is linear, constitutively non-local and with second order effects due to a state of initial stresses, it needs the satisfaction of 24 independent conditions at the boundaries. That is, since each of the 6 equations is of fourth order, consequently each polynomial of Eq. (29) should have four free constants, and for practical purposes they can be selected as the first four of each power series: $\bar{U}_{jk}^{*}$, $j = 1, ..., 6$, $k = 0, 1, 2, 3$. It is important to remark that the homogeneous boundary conditions must be 24 in order to have a well-possed eigenvalue problem. The 24 conditions have to be satisfied together with the differential equations in order to get the remaining coefficients $\bar{U}_{jk}^{*}$, $j = 1, ..., 6$, $k = 4, ..., Z$ of the power series in Eq. (29). Clearly, the eigenvalue problem has a determinant of order 24. Nevertheless, it is important to recall a useful aspect in the solution methodology of power series in the case of beam structures, that is, for the eigenvalue problem, the determinant can be reduced to a half of the original boundary conditions needed. In fact, the free coefficients of the equations in one boundary limit can be expressed in terms of the free coefficient of the other boundary limit, thus reducing the number of free coefficients to the half (Piovan et al., 2008).

In order to take advantage of this particular aspect, it is important to recall the steps of the recurrence to solve the problem (Piovan et al., 2008):

(1) Twelve of the 24 coefficients $\bar{U}_{jk}^{*}$, $j = 1, ..., 6$, $k = 0, 1, 2, 3$ are selected according to the boundary conditions. These twelve coefficients are denominated "free coefficients" and denoted as γ_i, $i = 1, ..., 12$.

(2) According to the linearity of the present problem one can appeal to the principle of superposition, performing alternatively the following twelve forms:

$$\left\{\gamma_i = 1, \gamma_j = 0, \forall j = 1, ..., 12; j \neq i\right\}, i = 1, ..., 12 \tag{30}$$

(3) After employing the twelve precedent alternatives in the corresponding boundary equations one arrives to:

$$\begin{bmatrix} \bar{B}_{11}\left(\bar{\Xi},\bar{\Omega}\right) & \cdots & \bar{B}_{112}\left(\bar{\Xi},\bar{\Omega}\right) \\ \vdots & \ddots & \vdots \\ \bar{B}_{121}\left(\bar{\Xi},\bar{\Omega}\right) & \cdots & \bar{B}_{1212}\left(\bar{\Xi},\bar{\Omega}\right) \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \vdots \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{31}$$

where: $\bar{\Xi}$ and $\bar{\Omega} = \lambda\Omega^2$ are the non-dimensional buckling and non-dimensional frequency eigenvalues, respectively, whereas $\bar{B}_{ij}$, $i, j = 1, ..., 12$ represents the $i - th$ boundary conditions that corresponds to the $j - th$ alternative mentioned in item (2). To calculate buckling eigenvalues, one settles $\bar{\Omega} = 0$, however in the case of frequency calculations $\bar{\Xi} \neq 0$ or $\bar{\Xi} = 0$, depending on the case to include or not the effect of initial stresses.

(4) Finally, if $\bar{\Omega}$ (or $\bar{\Xi}$) is the appropriate eigenvalue, then the following characteristic equation should be satisfied:

$$\det\left[\bar{B}_{ij}\left(\bar{\Xi},\bar{\Omega}\right)\right] = 0. \tag{32}$$

The recurrence scheme allows to shrink the algebraic problem from 6(Z+1) to only 12 unknown coefficients that can be selected according to the boundary equations. It has to be pointed out that the present power series scheme can be strongly simplified in some cases when decoupling is possible, as for example when no initial stresses or some initial stresses are incorporated in the problem or in the limit case of the beam in the context of local elasticity (i.e. $\bar{\mu} = 0$). This fact leads to the handling of two, three or four subsystems.

3.2 Finite element method

The kinematic variables of the problem can be discretized within the domain of a finite element by means of the following general form:

$$\mathbf{U} = \mathbf{N}\mathbf{q}_e \tag{33}$$

In which, $\mathbf{N}$ is a matrix whose rows ($\mathbf{N}_i$, $i = 1, ..., 6$) contains the shape functions for each kinematic variable. $\mathbf{q}_e$ is the vector of nodal variables. Then substituting Eq. (33) into Eq. (19), and assembling in the usual way the following finite element eigenvalues equations of the present nano-tube beam model is attained:

$$\left(\mathbf{K_E} + \bar{\Xi}\mathbf{K_G} - \bar{\Omega}^2\mathbf{M}\right)\bar{\mathbf{Q}} = \bar{\mathbf{O}}, \tag{34}$$

where $\mathbf{K_E}$, $\mathbf{K_G}$ and $\mathbf{M}$ are the global matrices of elastic stiffness, geometric stiffness and mass, respectively; whereas $\bar{\mathbf{Q}}$, is the global vector of nodal displacements and $\bar{\Xi}$ and $\bar{\Omega}$ are the non-dimensional buckling and frequency eigenvalues, respectively.

In this work, iso-parametric finite elements with 3 and 5 nodes are implemented and evaluated.

4 ILLUSTRATIVE EXAMPLES

4.1 Comparison of both approaches

In the following Table 1 the material and geometric properties of a single nano-tube that serves as campion for comparison purposes of the PSM and FEM developed in the previous paragraphs.

Table 1: Material and geometric properties of the carbon nano-beam.

Property Name	Value
Elasticity Modulus [GPa]	1100
Poisson Coefficient	0.25
Material Density [kg/m^3]	2100
Beam Length [nm]	10.00
Radius of the wall [nm]	1.00
Thickness of the wall [nm]	0.20

In Table 2 one can see the comparison between the Power series method and the finite element method, for a simply supported beam. Actually, the first five natural frequencies, in absence of initial stresses, are evaluated. In all calculations the following parameters have been employed: $Z = 50$ for the power series method and $N_e = 10$ ($N_e = 20$) for the finite element method in the case of 5-node (3-node) elements, which proved to be enough in order to satisfy a converge of less than 0.5% in the relative error up to the fifth eigenvalue.

Table 2: First five non-dimensional frequencies of carbon nano-beam in absence of initial stresses.

Method	$\bar{\mu}$	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$	$\bar{\Omega}_5$
Power Series	0	0.41137	0.41137	3.94784	4.57568	4.57568
FEM (5-Node)		0.41138	0.41138	3.94783	4.57574	4.57574
FEM (3-Node)		0.41140	0.41140	3.94792	4.57581	4.57581
Power Series	0.04	0.29494	0.29494	1.77412	1.77412	2.83043
FEM (5-Node)		0.29498	0.29498	1.77416	1.77416	2.83051
FEM (3-Node)		0.29502	0.29502	1.77435	1.77435	2.83068

It is possible to see the good matching of both methodologies. However, it is important to recall that the calculations have been carried out in the non-dimensional context, which is mandatory in the finite element method. This is important in order to avoid high matrix sparsity and/or numerical ill-posed matrices. On the other hand the power series method has shown a computational performance a little bit more expensive than the corresponding finite element approach, however it offers the possibility to reach quite accurate results.

The studies shown in the following paragraphs have been carried out with the finite element method (with 5-node elements) due to its computational versatility and faster calculation.

4.2 Buckling of nano-beams

In Table 3 one can see the variation of the non-dimensional buckling parameter with respect to the non-local parameter, subjected to an axial point load. Simply supported boundary conditions have been considered. For this case the buckling parameter is defined as $\bar{\Xi} = Q_x^0/J_{11}$

Table 3: Buckling parameters of the carbon nano-beam.

$\bar{\mu}$	$\bar{\Xi}$
0	4.223e-2
0.01	1.391e-3
0.09	1.547e-4
0.16	2.020e-5

Note that as the non-local parameter increases, the buckling parameter decreases significantly involving many orders of magnitude.

4.3 Vibrations of nano-beams

In Table 4 one can see the variation of the non-dimensional frequency parameter with respect to the non-local parameter, in absence of initial stresses. Clamped-clamped and clamped-free boundary conditions have been considered. In Table 5 one can see the influence of the normal initial stresses in the natural frequencies of a simply supported nano-beam. The calculation has been performed with a state of initial stresses parameterized with respect to the buckling load, i.e. a compressive state of initial stresses.

Table 4: First five non-dimensional frequencies in absence of initial stresses for clamped-clamped and clamped-free carbon nano-beams.

Bound. Condition	$\bar{\mu}$	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$	$\bar{\Omega}_5$
Clamped-clamped	0	1.42097	1.42097	3.94788	6.94142	6.94142
	0.01	1.27504	1.27504	3.59316	4.91161	4.91161
	0.09	0.69144	0.69144	1.44965	1.44965	1.44965
Clamped-free	0	0.05663	0.05663	0.98694	0.98694	1.49637
	0.01	0.05486	0.05486	0.96319	0.96319	1.18070
	0.09	0.04052	0.04052	0.45133	0.45133	0.80763

Table 5: Effect of initial stresses due to compression on the first natural frequency ($\bar{\Omega}_1$) of a simply supported carbon nano-beam.

$\bar{Q}_x^0/\bar{Q}_{xref}^0$	$\bar{\mu} = 0.04$	$\bar{\mu} = 0.09$
0.5	0.1472	0.1087
0.6	0.1177	0.0870
0.7	0.0883	0.0652
0.8	0.0589	0.0435
0.9	0.0294	0.0217

In both cases (with and without initial stresses), the effect of the non-local parameter is to diminish the value of the natural frequencies as the non-local parameter increases. However it is interesting to note that the influence of the non-local parameter is more pronounced in the case of the more constrained nano-beam. That is, if one compares in Table 4 the variation of the $\bar{\Omega}_1$ in the cases of $\bar{\mu} = 0.01$ to $\bar{\mu} = 0.09$ one observes a reduction in the frequency coefficient of a 45% and 26% in the clamped-clamped case and clamped-free case, respectively; whereas for $\bar{\Omega}_5$ it is observed reductions of 72% and 32% of the previous cases, respectively.

5 CONCLUSIONS

In this article a model of nano-beam for calculation of static and dynamic eigenvalues has been introduced. The model incorporates the non-local constitutive equations and couples extensional, bending and twisting motions. It also contemplates the presence of a state of initial normal stresses. This model can reproduce the response of the local elastic counterpart by eliminating the non-local effects, i.e. setting the non-local parameter to zero.

Two approaches have been developed in order to carry out with the calculation of static and dynamic eigenvalues: Power series method and finite element method. Both methods performed well and proved to be quite useful in the calculation of static and dynamic eigenvalues however due to the high scale factors, both had to be employed in non-dimensional form in order to avoid numerical singularities.

From the calculations carried out, the following particular conclusions can be mentioned: (a) The increase of the non-local parameter causes a decrease in the buckling load parameter. (b) In the vibration patterns, the effect of the non-local parameter is similar to the case of buckling, i.e. an increment of the non-local parameter causes the diminution of the natural frequencies, both in the presence of initial stresses or not; however it has been observed a more pronounced effect in the cases where the beams had more restrictive boundary conditions, for example the clamped-clamped case.

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