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REGULARIZATION OF ILL-POSED PROBLEMS WITH COMBINED QUADRATIC AND ANISOTROPIC BOUNDED-VARIATION PENALIZATION.

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Abstract. From the early works of Tikhonov and Phillips in 1962 and 1963, the treatment of inverse ill-posed problems has seen an enormous growth. Many methods, tools and ad-hoc algorithms bound to extract as much information as possible about the exact solution of the problem have been developed. In particular, during the last two decades a wide variety of new mathematical tools ranging from the use of variable L^p spaces to bounded-variation (BV) penalization, anisotropic diffusion methods and Bayesian models and hypermodels has arisen. Although it cannot be expected that a single method be better than all others for all type of problems, the ability of "detection" of discontinuities and borders and subsequent "self-adaptation" to different types of patterns, structures and degrees of regularity is a highly desired property of a regularization method. In this work we present some mathematical results on the existence and uniqueness of global minimizers of generalized Tikhonov-Phillips functionals with penalizers given by convex spatially-adaptive combinations of L^2 and isotropic and anisotropic BV type. Open problems are discussed and results to signal and image restoration problems are presented.

1 INTRODUCTION

A linear inverse problem can be formulated in the form: find u such that

$$Tu = v, \tag{1}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are two infinite dimensional normed spaces (usually Hilbert spaces of functions), T is a bounded linear operator with non-closed range between those two spaces, and v is the data, which is known or approximately known (with a certain error). In what follows, and unless we specify it differently, $\mathcal{X}$ will be $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded open convex set with Lipschitz boundary. Under the above hypotheses it is well known that problem (1) is ill-posed, the Moore-Penrose pseudo-inverse of $\mathcal{T}$ is unbounded and therefore small errors in the data v may result in arbitrarily large errors in the approximations of u (Spies and Temperini (2006)). For this reason, before attempting to solve problem (1), it must be "regularized". That means essentially replacing the problem by a family or sequence of "well-posed" problems whose solutions converge (in an appropriate way) to a solution of the original problem (1). Undoubtedly, the most usual way of regularizing a linear ill-posed problem is by means of the Tikhonov-Phillips method, which can be formulated in a few different ways. First within a general mathematical theory by means of spectral theory (see Engl et al. (1996)) but also as a simple unconstrained minimization problem. In fact, the regularized solution obtained by the Tikhonov-Phillips method and a penalizer W with domain $\mathcal{D} \subset \mathcal{X}$, is the global minimizer over $\mathcal{D}$ (provided such a minimizer exists), of the functional

$$J_{\alpha,W}(u) = \|Tu - v\|^2 + \alpha W(u), \tag{2}$$

where $\alpha > 0$ is a constant called regularization parameter. The original method was proposed independently by Phillips and Tikhonov in 1962 and 1963 (Phillips (1962), Tikhonov (1963a), Tikhonov (1963b)) using $W(u) = \|u\|^2_{\mathcal{X}}$. Other penalizers can also be used to regularize the problem and in the last two decades, considerable research has been devoted studying what types of functionals can be used for that purpose and, given a problem, decide which one is more appropriate to preserve certain known properties of the exact solution. Thus, for instance, choosing $W(u) = \|u\|^2_{\mathcal{X}}$ results always smooth regularized approximations which converge, as $\alpha \to 0^+$, to the so called "best approximate solution" (i.e. the least squares solution of minimum norm) of problem (1) (see Engl et al. (1996)) while $W(u) = \| |\nabla u| \|^2_{L^2(\Omega)}$ corresponds to the order-one Tikhonov-Phillips method. On the other hand, $W(u) = \|u\|_{\mathrm{BV}(\Omega)}$ (where $\|\cdot\|_{\mathrm{BV}}$ denotes the total variation norm) or $W(u) = \| |\nabla u| \|_{L^1(\Omega)}$, result in the so called "bounded variation regularization methods" (Acar and Vogel (1994), Rudin et al. (1992)) which are strongly suggested when preserving discontinuities or edges that could be present in the exact solution is an important matter. These methods, however, tend to produce piecewise constant approximations and therefore they will most likely be inappropriate in regions where the exact solution is smooth (Chambolle and Lions (1997)), producing the so called "staircasing effect". For general penalizers W, sufficient conditions guaranteeing existence, uniqueness and stability of the minimizers under different types of perturbations were found in Mazzieri et al. (2012). There are several reasons why it is important to use an appropriate penalizer for regularizing a problem. The main one being that the penalizing term engraves the approximate solution with particular properties which one believes, or one has good motives to believe, that the exact solution also possesses. With that in mind it is reasonable to think that the use of two or more penalizers of different nature, that could somehow spatially adapt to the local characteristics of the exact solution, could be more convenient. During the last 15 years many regularization methods have been developed in light of this line of reasoning. Thus, for instance, in 1997 Blomgren *et al.* (Blomgren et al. (1997)) proposed the following penalizer:

$$W(u) = \int_\Omega |\nabla u|^{p(|\nabla u|)} dx, \tag{3}$$

where p is a decreasing function satisfying $\lim_{u\to 0^+} p(u) = 2$, $\lim_{u\to\infty} p(u) = 1$. Thus, in regions where the gradient of u is small the penalizer approximates $\||\nabla u|\|^2_{L^2(\Omega)}$, what corresponds to a Tikhonov-Phillips method of order one (appropriate for smooth regions) while for the gradient of u large, the penalizer resembles the bounded variation seminorm $\||\nabla u|\|_{L^1(\Omega)}$, which, as previously mentioned, is appropriate for border detection purposes. Although this model for W is quite reasonable, proving basic properties of the corresponding generalized Tikhonov-Phillips functional turns out to be quite difficult. Existence of global minimizers of functional (2) with W given by (3), was proved by the authors by using the theory of variable L^p spaces. Also in 1997 Chambolle and Lions suggested a different way of combining these two methods (Chambolle and Lions (1997)) by defining a thresholded penalizer of the form:

$$W_\beta(u) = \int_{|\nabla u|\leq\beta} |\nabla u|^2\, dx + \int_{|\nabla u|>\beta} |\nabla u|\, dx,$$

where $\beta > 0$ is a prescribed threshold parameter. Thus, in regions where borders are more likely to be present ($|\nabla u| > \beta$), penalization is made with the bounded variation seminorm while a standard order-one Tikhonov-Phillips method is used otherwise. This model was shown to be successful in restoring images possessing regions with homogeneous intensity separated by borders. However, in the case of images with non-uniform or highly degraded intensities, the model is extremely sensitive to the choice of the threshold parameter β. More recently, penalizers of the form

$$W(u) = \int_\Omega |\nabla u|^{p(x)} dx, \tag{4}$$

for certain functions p with range in $[1, 2]$, were studied in Chen et al. (2006) and Li et al. (2010). However, it is timely to point out here that all previously mentioned results are valid only for the case of denoising (no blurring), i.e. for the case $T = id$.

In this article we will study penalizers of the form

$$W(u) = \alpha_1 \int_\Omega |\sqrt{1-\theta(x)}\, u(x)|^2\, dx + \alpha_2 \int_\Omega \|\theta(x)A(x)\nabla u(x)\|\, dx \tag{5}$$

where α_1, α_2 are positive constants, $\theta(x)$ is a weighting function with values on the interval $[0, 1]$ and $A(x)$ is a symmetric positive definite matrix field. Existence, uniqueness and stability results of global minimizers of the corresponding generalized Tikhonov-Phillips functionals will be derived. Several remarks are in order. Some of these results are anisotropic generalizations of similar results obtained in Mazzieri et al. (2014a). Note that the extreme case $\theta(x) = 0\ \forall x$ corresponds to the classical Tikhonov-Phillips method. For $\theta(x) = 1\ \forall x$ one gets a pure BV method, with the classical Bounded Variation method corresponding to the case of $A(x) = I\ \forall x$. Other choices of the matrix field A are possible in order to induce an anisotropic BV penalization. Feasible ways of constructing this matrix field can be found for instance in Calvetti et al. (2006). The general case can then be thought of as a convex combination of a classical L^2 and an anisotropic BV penalizers.

2 MAIN RESULTS

In this section we will state our main results concerning existence and uniqueness of minimizers of particular generalized Tikhonov-Phillips functionals with combined L^2-BV penalizers. Due to brevity and since complete proofs of these results will appear in a forthcoming paper, we will include no the proofs here, limiting a somewhat deeper discussion only to those considered more relevant.

In what follows Ω shall denote a bounded open convex subset of $\mathbb{R}^n$ with Lipschitz boundary, $\mathcal{M}(\Omega)$ denotes the set of all real valued measurable functions defined on Ω and $\widehat{\mathcal{M}}(\Omega)$ the subset of $\mathcal{M}(\Omega)$ consisting of those functions with values in $[0, 1]$.

2.1 The isotropic case

Definition 2.1 *Given $\theta \in \widehat{\mathcal{M}}(\Omega)$, we define the functional $W_{0,\theta}(u)$ with values on the extended reals by*

$$W_{0,\theta}(u) \doteq \sup_{\vec{\nu}\in\mathcal{V}_\theta} \int_\Omega -u(x)\, \mathbf{div}(\theta(x)\vec{\nu}(x))\, dx, \qquad u \in \mathcal{M}(\Omega) \tag{6}$$

where $\mathcal{V}_\theta \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^n$ such that $\theta\vec{\nu} \in C_0^1(\Omega)$ and $|\vec{\nu}(x)| \le 1 \,\forall\, x \in \Omega\}$ and $|\cdot|$ denotes the euclidean 2-norm in $\mathbb{R}^n$.

It is important to point out that if u and θ are both in $C^1(\Omega)$, then it can be proved that $W_{0,\theta}(u) = \|\theta\,|\nabla u|\,\|_{L^1(\Omega)}$.

In Mazzieri et al. (2013) a result regarding existence and uniqueness of global minimizers of the functional

$$F_\theta(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2\, W_{0,\theta}(u), \qquad u \in L^2(\Omega) \tag{7}$$

was proved. However, the proof of such a result (see Theorem 3.4 in Mazzieri et al. (2013)) precludes the case in which θ assumes the extreme values 0 or 1 on a set of positive measure. It is timely to point out nonetheless that both cases are of practical interest since in some cases a pure BV regularization in some regions and a pure L^2 regularization in others may be desired. The next three theorems deal precisely with results about existence and uniqueness of minimizers of functional (7) for the cases in which the weighting function θ can take the extreme values 0 or 1 on sets of positive measure. Complete details on the proofs of the following theorems can be found in (Mazzieri et al., 2014a).

Theorem 2.2 *Let $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a normed vector space, $v \in \mathcal{Y}$, α_1, α_2 positive constants, $\theta \in \widehat{\mathcal{M}}(\Omega)$ and $\Omega_0 \doteq \{x \in \Omega$ such that $\theta(x) = 0\}$. If $\frac{1}{\theta} \in L^\infty(\Omega_0^c)$ and $\frac{1}{1-\theta} \in L^1(\Omega_0^c)$ then functional (7) has a unique global minimizer $u^* \in L^2(\Omega) \cap BV(\Omega_0^c)$.*

Theorem 2.3 *Let $\mathcal{X}$, $\mathcal{Y}$, v, α_1, α_2 as in Theorem 2.2. Assume further that $\mathcal{Y}$ is a Hilbert space, let $\theta \in \widehat{\mathcal{M}}(\Omega)$ and $\Omega_1 \doteq \{x \in \Omega$ such that $\theta(x) = 1\}$. If $n \le 2$, $\frac{1}{\theta} \in L^\infty(\Omega_1^c)$, $\frac{1}{1-\theta} \in L^1(\Omega_1^c)$ and $T\chi_\Omega \ne 0$, then functional (7) has a global minimizer $u^* \in L^2(\Omega) \cap BV(\Omega_1^c)$. If moreover T is injective, then such a global minimizer is unique.*

Theorem 2.4 *Let n, $\mathcal{X}$, $\mathcal{Y}$, v, α_1, α_2, Ω_1 as in Theorem 2.3, Ω_0 as in Theorem 2.2 and $\theta \in \widehat{\mathcal{M}}(\Omega)$. If $\frac{1}{\theta} \in L^\infty(\Omega_0^c)$, $\frac{1}{1-\theta} \in L^\infty(\Omega_1^c)$ and T is injective, then functional (7) has a unique global minimizer $u^* \in L^2(\Omega) \cap BV(\Omega_1^c \cap \Omega_0^c)$.*

2.2 The anisotropic case

In this section we will state a few results regarding the case in which the penalizer $W_{0,\theta}$ in (6) is modified to take into account anisotropic $BV-$diffusion. This will be achieved by means of an appropriately constructed matrix field A. The construction of this matrix field is a very important matter on which we shall not get any deeper here. We will only mention that there are several ways of constructing this so-called "anisotropic matrix field", either from structural prior information or from the available data (see Kaipio et al. (1999), Calvetti et al. (2006), Grasmair and Lenzen (2010)).

Definition 2.5 *Given $\theta \in \widehat{\mathcal{M}}(\Omega)$ and a measurable matrix field $A : \Omega \to \mathbb{R}^{2\times 2}$ we define the functional $W_{\theta,A}(u)$ with values on the extended reals by*

$$W_{\theta,A}(u) \doteq \sup_{\vec{\nu}\in\mathcal{V}_{\theta,A}} \int_\Omega -u(x)\, div(\theta(x)A(x)\vec{\nu}(x))\, dx, \qquad u \in \mathcal{M}(\Omega) \tag{8}$$

where $\mathcal{V}_{\theta,A} \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^2$ such that $\theta A\vec{\nu} \in C^1_0(\Omega)$ and $|\vec{\nu}(x)| \leq 1\, \forall\, x \in \Omega\}$.

Just like in the isotropic case, it can be proved that if u, $\theta \in C^1(\Omega)$ and $A \in C^1(\Omega, \mathbb{R}^{2\times 2})$ is a symmetric measurable matrix field, then $W_{\theta,A}(u) = \|\theta\, |A\nabla u|\, \|_{L^1(\Omega)}$.

Consider the functional

$$F_{\theta,A}(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2\, W_{\theta,A}(u), \qquad u \in L^2(\Omega). \tag{9}$$

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a normed vector space, $v \in \mathcal{Y}$, α_1, α_2 positive constants, $\theta \in \widehat{\mathcal{M}}(\Omega)$ such that $\frac{1}{1-\theta} \in L^1(\Omega)$ and $\frac{1}{\theta} \in L^\infty(\Omega)$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that $\|A^{-1}(x)\| \leq 1\, \forall\, x \in \Omega$. Then functional $F_{\theta,A}$ defined by (9) has a unique global minimizer $u^* \in BV(\Omega)$.*

Remark 2.7 *Note that if $\theta(x) = 0\ \forall\, x \in \Omega$, then $F_{\theta,A}$ as defined in (9) is the classical Tikhonov-Phillips functional of order zero while for $\theta(x) = 1\ \forall\, x \in \Omega$ a pure anisotropic BV penalty is obtained. Although the hypotheses of Theorem 2.6 clearly excludes the later case, the next theorem provides conditions for existence and uniqueness of a global minimizer of (9) when $\theta(x) = 1\ \forall\, x \in \Omega$.*

Theorem 2.8 *(Anisotropic BV) Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a Hilbert space, $v \in \mathcal{Y}$, α a positive constant, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that $\|A^{-1}(x)\| \leq 1\, \forall\, x \in \Omega$ and assume $T\chi_\Omega \neq 0$. Then the functional*

$$F_A(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha \sup_{\vec{\nu}\in\mathcal{V}_A} \int_\Omega -u(x)\, div(A(x)\vec{\nu}(x))\, dx, \tag{10}$$

where $\mathcal{V}_A \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^2$ such that $A\vec{\nu} \in C^1_0(\Omega)$ and $|\vec{\nu}(x)| \leq 1\, \forall\, x \in \Omega\}$, has a global minimizer $u^* \in L^2(\Omega)$. Moreover, if T is injective then such a global minimizer is unique.

Currently we also have partial results for this anisotropic situation in cases in which the weighting function θ can take the extreme values 0 or 1 on sets of positive measure. This research is currently underway and detailed results will be published in a forthcoming article (Mazzieri et al. (2014b)).

3 NUMERICAL RESULTS

The purpose of this section is to show some applications of the regularization methods presented in the previous section consisting in the simultaneous use of penalizers of L^2 and of bounded-variation (BV) type to signal and image restoration problems.

3.1 APPLICATIONS TO SIGNAL RESTORATION

A basic mathematical model for signal blurring is given by convolution, as a Fredholm integral equation of first kind:

$$v(t) = \int_0^1 k(t,s)u(t)ds, \tag{11}$$

where $k(t,s)$ is the blurring kernel or point spread function, u is the exact (unknown) signal and v is the blurred signal. For the examples that follow we took a Gaussian blurring kernel, i.e. $k(t,s) = \frac{1}{\sqrt{2\pi}\sigma_b} \exp\left(-\frac{(t-s)^2}{2\sigma_b^2}\right)$, with $\sigma_b > 0$. Equation (11) was discretized in the usual way (using collocation and quadrature), resulting in a discrete model

$$Af = g, \tag{12}$$

where A is a $(n+1)\times(n+1)$ matrix, $f, g \in \mathbb{R}^{n+1}$ $(f_j = u(t_j),\ g_j = v(t_j),\ t_j = \frac{j}{n},\ 0 \le j \le n)$. We took $n = 130$ and $\sigma_b = 0.05$. The data g was contaminated with a 1% zero-mean Gaussian additive noise (i.e. standard deviation equal to 1% of the range of g).

We considered a signal which is smooth in two disjoint intervals and piecewise constant in the complement of their union, having three finite jumps. The signal was blurred and noise was added as described. The original and blurred-noisy signal are depicted in Figure 1.

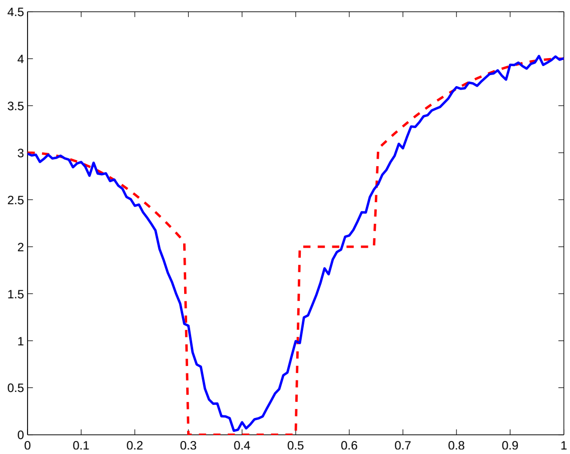

Figure 1: Original (- -) and blurred-noisy (—) signals.

Figure 2 shows the restorations obtained with the classical zero-order Tikhonov-Phillips method (left) and with a pure $BV-$penalizer (i.e. (7) with $\theta(x) = 1\,\forall x \in \Omega$) (right). In all cases the regularization parameters were optimally chosen. As expected, the regularized solution obtained with the later method is significantly better than the one obtained with the classical Tikhonov-Phillips method near jumps and in regions where the exact solution is piecewise constant. The opposite happens where the exact solution is smooth.

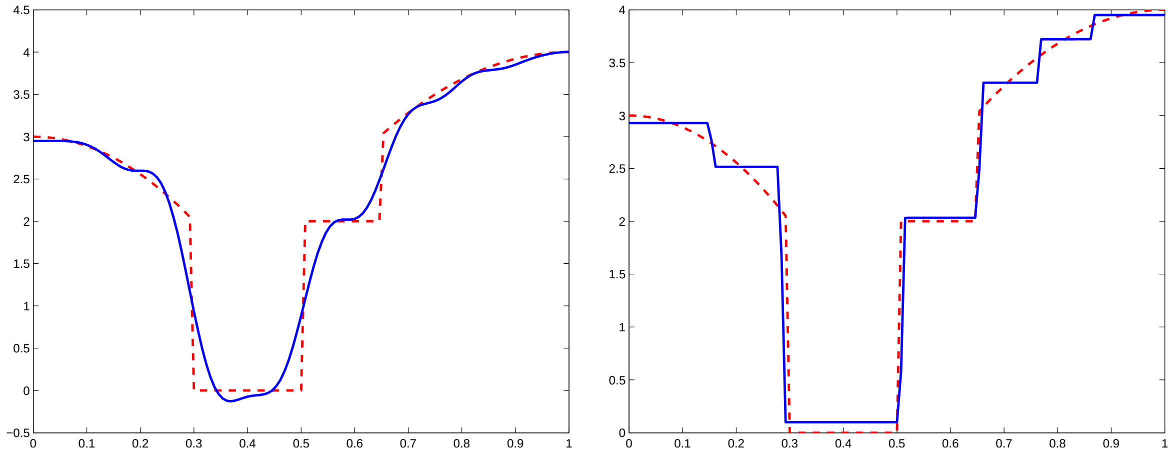

Figure 2: Original signal (- -) and regularized solutions (—) obtained with Tikhonov-Phillips (left) and bounded variation seminorm (right).

An ad-hoc binary weight function theta for this example was defined on the interval $[0, 1]$ as $\theta(t) = 1$ for $t \in [0.3,\ 0.65]$ and $\theta(t) = 0$ for $t \in [0,\ 0.3) \cup (0.65,\ 1]$. The regularized solution obtained with this weight function and the combined L^2-BV method is shown in Figure 3. The improvement with respect to any of the single classical pure methods is clearly notorious.

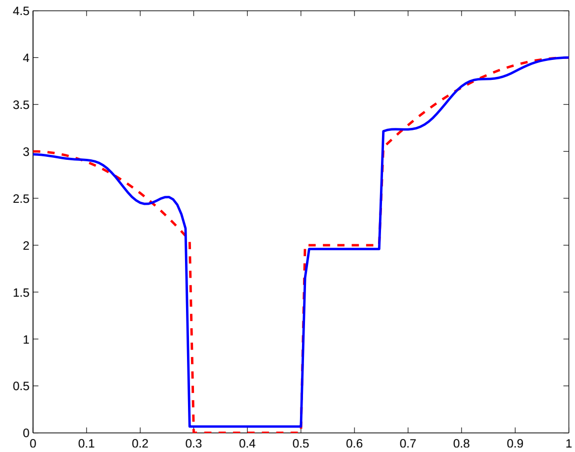

Figure 3: Original signal (- -) and regularized solution (—) obtained with the combined L^2-BV method and an ad-hoc binary function θ.

Although this choice of $\theta(t)$ is clearly based upon "*a-priori*" information about the regularity of the exact solution, other reasonable choices of θ can be made by using only data-based information. For instance, one way of constructing a reasonable weighting function θ is by computing the normalized (in $[0, 1]$) convolution of a zero-mean Gaussian function with standard deviation σ_b and the modulus of the gradient of the regularized solution obtained with a pure zero-order Tikhonov-Phillips method (see Figure 4). For this function θ, the corresponding regularized solution obtained with the combined L^2-BV method is shown in Figure 5. In all cases reflexive boundary conditions were used (Hansen (2010)) and the optimal regularization parameters were estimated using Morozov's discrepancy principle with $\tau = 1.1$ (Engl et al. (1996)).

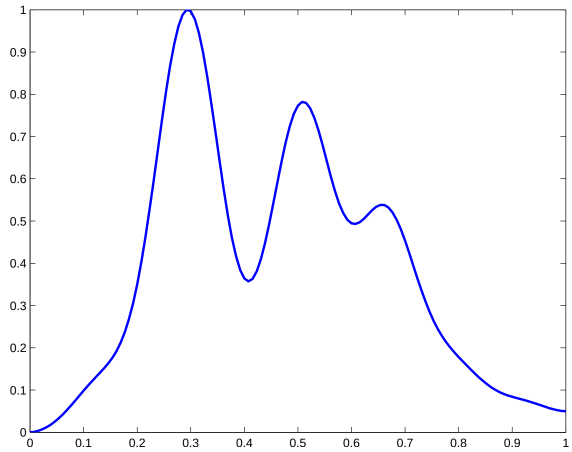

Figure 4: Tikhonov-based weight function θ.

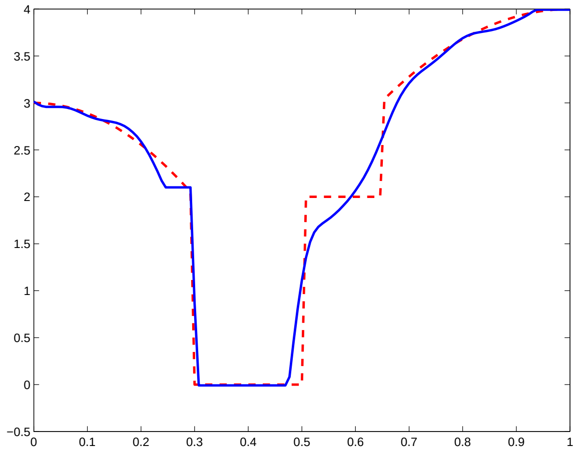

Figure 5: Original signal (- -) and regularized solution (—) obtained with the combined L^2-BV method and function θ showed in Fig. 4.

Table 1 shows the ISNR values of the four performed restorations. These values show once again a significant improvement of the combined method with respect to any of the pure (single) methods.

Regularization Method	**ISNR**
Tikhonov-Phillips	2.6008
Bounded variation (BV)	2.8448
Mixed L^2-BV with binary θ	4.8969
Mixed L^2-BV with Tikhonov-based θ	4.3315

Table 1: ISNR values of the different restorations

3.2 APPLICATIONS TO IMAGE RESTORATION

For this case the forward blurring model is given by convolution with a point spread function of “atmospheric turbulence” type, i.e., with a two-dimensional Gaussian function with horizontal and vertical standard deviations σ_h and σ_v, respectively. Data for the inverse problem is then obtained by adding to the blurred image, a zero-mean gaussian noise with $\sigma\,\%$ standard deviation. Figure 6 (a) shows the blurred noisy image corresponding to $\sigma_h = \sigma_v = 0.015$ and $\sigma = 2$ while Figure 6 (b) contains the restoration obtained with a Tikhonov-Phillips method (pure L^2 penalizer). This restoration was later used to build the anisotropic penalization matrix field A and the weighting function θ of the mixed regularization L^2-BV.

(a) (b)

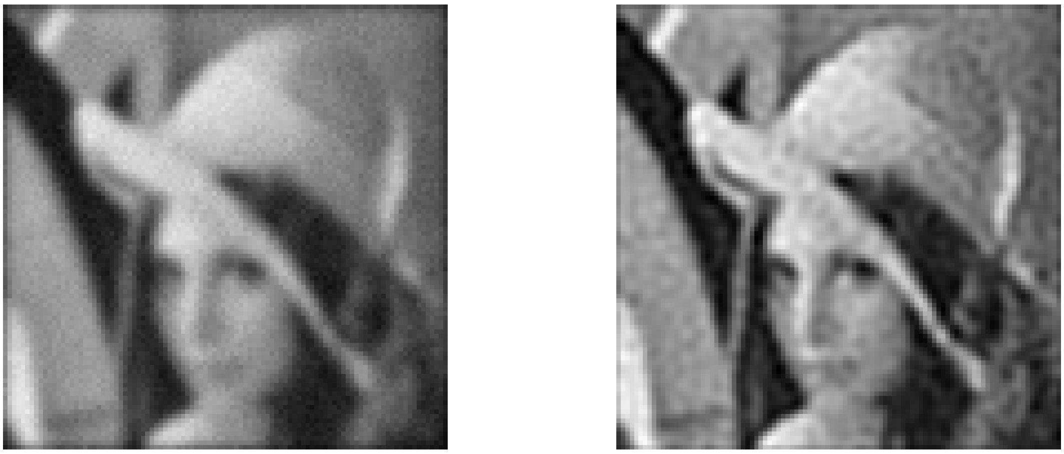

Figure 6: (a) Blurred noisy image (observation); (b) Tikhonov-Phillips restoration.

Figure 7 shows the restorations obtained using pure BV penalizers; isotropic case in (a) and anisotropic in (b). It is clear to observe the better performance of the anisotropic BV method with respect to isotropic one, particularly in regard to edge detection. This improvement is also reflected in the ISNR, which are presented in Table 2.

(a) (b)

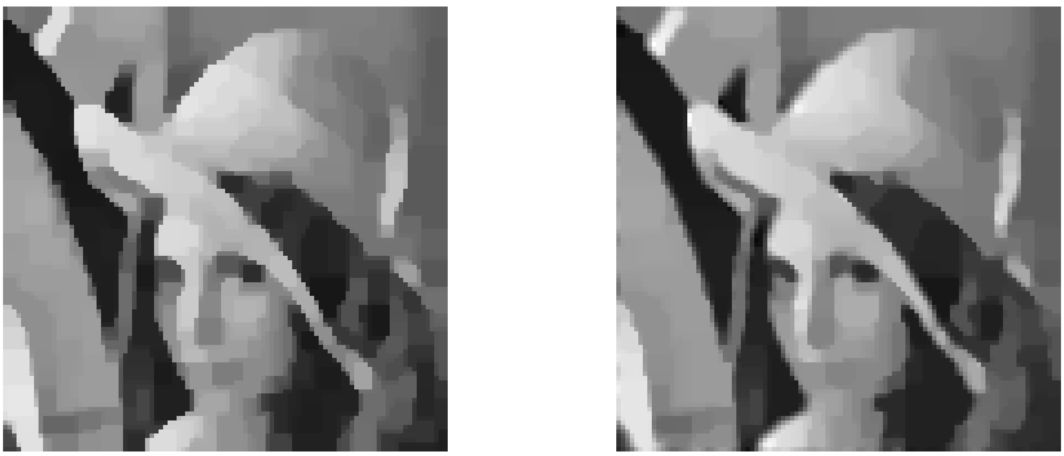

Figure 7: (a) Isotropic BV restoration; (b) Anisotropic BV restoration.

Figure 8 shows the restoration images obtained with the new mixed L^2-BV penalizer; isotropic case in (a) and anisotropic in (b). In both cases, the weighting function θ was computed in the same way as previously described for signal restoration. The construction of the anisotropic penalization matrix field A was realized following the steps given in Calvetti et al. (2006).

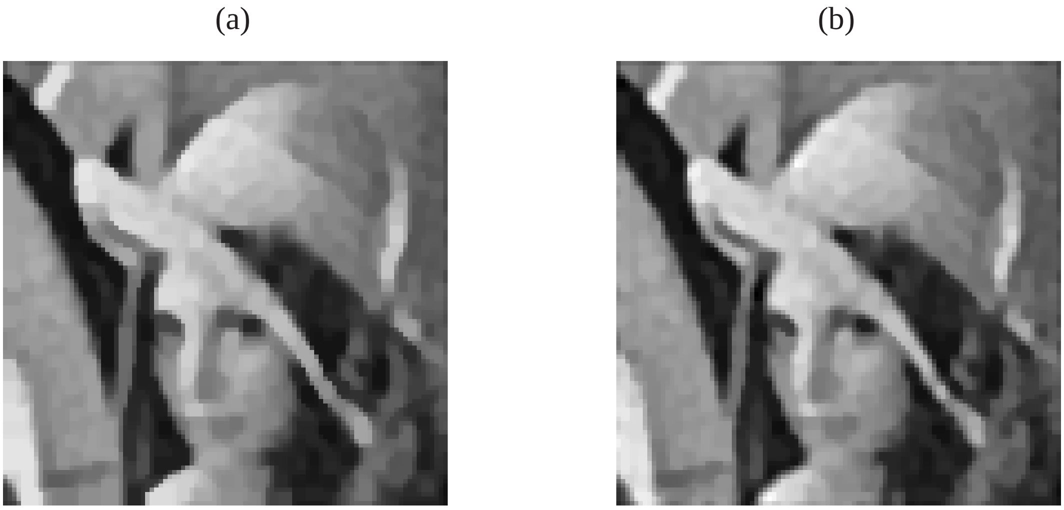

Figure 8: (a) Mixed L^2-isotropic BV restoration; (b) Mixed L^2-anisotropic BV restoration.

The original image is presented in Figure 9. The ISNR value was computed in order to have an objective parameter to measure and compare the quality of all image restorations (see Table 2). These values show a improvement of the both combined methods, isotropic and anisotropic, with respect to any of the pure methods. In turn, it is important to note that the incorporation of regularity information of the exact solution through the anisotropic penalization matrix field A yields better results with respect to the combined isotropic method.

Figure 9: Original image

Regularization Method	**ISNR**
Tikhonov-Phillips	3.166
Isotropic BV	2.745
Anisotropic BV	3.343
Mixed Isotropic	3.403
Mixed Anisotropic	3.844

Table 2: ISNRs of each restoration

4 CONCLUSIONS

In this work we presented several mathematical results on the existence and uniqueness of global minimizers of generalized Tikhonov-Phillips functionals with penalizers given by convex spatially-adaptive combinations of L^2 and isotropic and anisotropic BV type. Open problems were discussed and results to signal and image restoration problems were presented. These results are consistent with the intuitive foundations upon which the new methods are based, and they show a significant improvement in their performance with respect to the traditional methods.

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