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APPROXIMATE SOLUTIONS TO FREDHOLM INTEGRAL EQUATIONS BY ITERATED PETROV-GALERKIN METHOD WITH REGULAR PAIRS

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Abstract. In this work we use Petrov-Galerkin method for solving classical Fredholm integral equations of the second kind. The notion of regular pair of finite dimensional subspaces – simply characterized by the positive definiteness of a correlation matrix - makes it easy to guarantee solvability and numerical stability of the approximation scheme. In addition, by an iteration of the method the approximate solution can be improved. We show how the error is reduced by means of this procedure and build a better approximation.

1 INTRODUCTION

Classical Fredholm equations of the second kind are integral equations of the form

$$u(t)-\int_a^b k(s,t)u(s)ds = g(t) \qquad \forall t\in[a,b] \tag{1}$$

where u is the unknown function to be determined in a Banach space X while the kernel $k:[a,b]\times[a,b]\to R$ and the right-hand side $g:[a,b]\to R$ are given functions.

If we introduce the operator $K: X\to X$, with

$$(Ku)(t)=\int_a^b k(s,t)u(s)ds\,,$$

we can rewrite the equation as

$$u-Ku=g\,. \tag{2}$$

If $k:[a,b]\times[a,b]\to R$ is a continuous function, K is a linear compact operator and a sufficient condition to guarantee the existence and uniqueness of a solution of (2) is $\|K\|<1$ (see Kress, 2014, p. 23).

Petrov-Galerkin method has been proposed to find numerical approximate solutions to this type of integral equations, projecting on two sequences of appropriate finite dimensional subspaces of X, $X_n = span\{\varphi_j^n, j=1,\ldots,d_n\}\subset X$ and $Y_n = span\{\psi_i^n, i=1,\ldots,d_n\}\subset X$, named the trial and test subspaces respectively. In the case of being X a Hilbert space with inner product $\langle\,.\,\rangle$, Petrov-Galerkin method looks for $u_n\in X_n$ such that

$$\langle u_n-Ku_n, v\rangle = \langle g, v\rangle \qquad \forall v\in Y_n \tag{3}$$

and, as X_n and Y_n are finite dimension subspaces, solving (3) reduces to find the solution to a system of linear equations represented by a $d_n\times d_n$ matrix B:

$$\sum_{j=1}^{d_n}\underbrace{\left[\langle\varphi_j^n,\psi_i^n\rangle-\langle K\varphi_j^n,\psi_i^n\rangle\right]}_{B_{ij}}c_j^n=\langle g,\psi_i^n\rangle \qquad \forall i=1,\ldots,d_n \tag{4}$$

In Chen and Xu (1998) it has been proved that, if $K: X\to X$ is a compact linear operator not having 1 as an eigenvalue and the pair $\{X_n, Y_n\}$ is a *regular pair* - in a sense that will be detailed in the next section - equation (3) has a unique solution u_n that satisfies

$$\|u-u_n\|\le C\inf_{x\in X_n}\|x-u\| \tag{5}$$

for u the exact solution of (2).

Solvability and numerical stability of the approximation scheme are guaranteed and, as can be noted in (5), the accuracy of the approximation u_n to the unique solution u of (2) does not depend formally on Y_n. Then, the idea is to build regular pairs $\{X_n, Y_n\}$ with test functions subspaces, Y_n, easy to handle, knowing that good convergence of the method is preserved. In this way, the calculations are simple and the convergence is good.

But the approximations can be even improved by means of an iteration of the method. As it is shown in Chandler (1979) and Sloan (1976), once $u_n \in X_n$ are obtained, a new sequence of approximate solutions, $u_n^* \in X_n$, can be built by means of a simple procedure,

$$u_n^* = g + Ku_n. \tag{6}$$

In this work we first choose pairs of subspaces $\{X_n, Y_n\}$ generated by Legendre polynomials; we prove they are regular pairs for every $n \in N$ and show the goodness of the of approximations in a numerical example with known solution. Then, following Sloan (1976), we improved the convergence by means of an iteration of the method and showed, following Chandler (1979), why the approximation is better than the one obtained before iterating, even for small values of $n \in N$.

2 PETROV-GALERKIN METHOD WITH REGULAR PAIRS

Consider a Hilbert space $(X, \langle\,.\,\rangle)$, $\|\ \|$ the associated norm and $K: X \to X$ a compact linear operator not having 1 as an eigenvalue. It is shown in Kress (2014) that, under these conditions, if there exists any solution $u \in X$ to equation (2),with $g \in X$ a given function, it is unique. We are interested in looking for this solution $u \in X$ or, at least, a *good* approximation of it.

Choosing, for each $n \in N$, subspaces $X_n \subset X, Y_n \subset X$ of finite dimensions, with $\dim(X_n) = \dim(Y_n)$, Petrov-Galerkin method for equation (2) is a numerical method for finding $u_n \in X_n$ satisfying (3). The goal is to establish conditions under which (3) has a unique solution $u_n \in X_n$ and $\lim_{n\to\infty} \|u_n - u\| = 0$, for u satisfying (2).

In Chen and Xu (1998) it is proved that a condition ensuring the existence of a unique $u_n \in X_n$ verifying (3) is

$$X_n^{\perp} \cap Y_n = \{0\}. \tag{7}$$

Related to the condition of convergence, from Kress (2014) we can expect it only if, for each $x \in X$, there exists a sequence $\{x_n, n \in N\} \subset X$ with $\lim_{n\to\infty} x_n = x$; so, from now on, the families of subspaces $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$ are both chosen fulfilling this condition of denseness.

Denoting $\{X_n, Y_n\}$ the sequences of subspaces, according to Chen and Xu (1998), $\{X_n, Y_n\}$ is said to be a *regular pair* if there exists a linear surjective operator $\prod_n : X_n \to Y_n$ so that:

i) $\exists C_1 \in R \,/\, \forall x \in X_n : \|x\| \le C_1 \sqrt{\langle x, \prod_n x\rangle}$

ii) $\exists C_2 \in R \,/\, \forall x \in X_n : \|\prod_n x\| \le C_2 \|x\|$

It is easy to show that surjectivity of $\prod_n$ and i) guarantee (7), and the following theorem from Chen and Xu (1998, p. 411) summarizes the conditions for the existence and uniqueness of the solutions of (3) and their convergence to the solution of (2):

Theorem 2.1: Let X be a Banach space and $K: X \to X$ *a compact linear operator not having 1 as eigenvalue. Suppose* X_n *and* Y_n *are finite dimensional subspaces of X, with* $\dim(X_n) = \dim(Y_n)$, *verifying that* $\{X_n, Y_n\}$ *is a regular pair and, for each* $x \in X$, *there exist sequences* $\{x_n, n \in N\} \subset X_n$ *and* $\{y_n, n \in N\} \subset Y_n$ *so that* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = x$ *Then, there exists* $n_0 \in N$ *such that, for* $n > n_0$, *equation* $\langle u_n - Ku_n, v\rangle = \langle g, v\rangle \quad \forall v \in Y_n$ *has a unique solution* $u_n \in X_n$ *for any given* $g \in X$, *that satisfies* $\|u - u_n\| \le C \inf_{x \in X_n} \|x - u\|$, *where* $u \in X$ *is the unique solution of* $u - Ku = g$ *and C is constant not dependent of n.*

From Chen, Micchelli and Xu (1997), characterization of regular pairs is easy by means of the so called "correlation matrix", as follows.

If $\{\varphi_j^n, j = 1,\ldots,d_n\}$ and $\{\psi_i^n, i = 1,\ldots,d_n\}$ are basis of X_n and Y_n respectively, the following $d_n \times d_n$ matrices are defined: $G(X_n) := \left[\langle \varphi_i^n, \varphi_j^n \rangle\right]$, $G(Y_n) := \left[\langle \psi_i^n, \psi_j^n \rangle\right]$, $G(X_n, Y_n) := \left[\langle \varphi_i^n, \psi_j^n \rangle\right]$ and $G_+(X_n, Y_n) := \frac{1}{2}[G(X_n, Y_n) + G(Y_n, X_n)]$. $G(X_n)$ and $G(Y_n)$ are positive definite, as they are the gramian matrices of the inner product in X_n and Y_n respectively; $G(X_n, Y_n)$ is the "correlation matrix" and $G_+(X_n, Y_n)$ is its symmetric part. Following Chen, Micchelli and Xu (1997) we prove (Orellana Castillo, Seminara and Troparevsky, 2018) the following

Proposition 2.1: If $G_+(X_n, Y_n)$ *is a positive definite matrix,* $\{X_n, Y_n\}$ *is a regular pair.*

Proof:

As $G(X_n)$ and $G(Y_n)$ are positive definite, $0 < \lambda_{\min}\|x\|^2 \le x^T G(X_n) x \le \lambda_{\max}\|x\|^2$ $\forall x \in X_n, x \ne 0$ and $0 < \nu_{\min}\|x\|^2 \le x^T G(Y_n) x \le \nu_{\max}\|x\|^2 \quad \forall x \in X_n, x \ne 0$, for $\lambda_{\min}$ and $\lambda_{\max}$ the minimum and maximum (real) eigenvalues of $G(X_n)$ and $\nu_{\min}$ and $\nu_{\max}$ the minimum and maximum (real) eigenvalues of $G(Y_n)$.

Similarly, if $G_+(X_n, Y_n)$ is assumed to be positive definite and $\mu_{\min}$ and $\mu_{\max}$ are its minimum and maximum eigenvalues, it is $0 < \mu_{\min}\|x\|^2 \le x^T G_+(X_n, Y_n) x \le \mu_{\max}\|x\|^2$ $\forall x \in X_n, x \ne 0$.

Now, for $\sigma = \frac{\lambda_{\max}}{\mu_{\min}}$ and $\sigma' = \frac{\nu_{\max}}{\lambda_{\min}}$, we have $0 < x^T G(X_n) x \le \sigma\, x^T G_+(X_n, Y_n) x \quad \forall x \in X_n, x \ne 0$ and $0 < x^T G(Y_n) x \le \sigma'\, x^T G_+(X_n, Y_n) x \quad \forall x \in X_n, x \ne 0$.

Defining $\Pi_n : X_n \to Y_n \,/\, \Pi_n(\varphi_i^n) = \psi_i^n$ for $i = 1,2,\ldots,d_n$, it is clearly surjective and, for $x = \sum_{i=1}^{d_n} \alpha_i \varphi_i^n$, we have

$$\|x\|^2 = \alpha^T G(X_n)\alpha \le \sigma\, \alpha^T G_+(X_n, Y_n)\alpha = \sigma\, \alpha^T \tfrac{1}{2}[G(X_n, Y_n) + G(Y_n, X_n)]\alpha =$$

$$= \sigma \left\langle x, \sum_{i=1}^{d_n} \alpha_i \psi_i^n \right\rangle = \sigma \langle x, \Pi_n(x) \rangle$$

Additionally, $\|\Pi_n(x)\|^2 = \alpha^T G(Y_n)\alpha \le \sigma'\, \alpha^T G(X_n)\alpha = \sigma' \|x\|^2$, and the conditions for $\{X_n, Y_n\}$ to be a regular pair are fulfilled, with $C_1 = \sqrt{\sigma}$ .and $C_2 = \sqrt{\sigma'}$.◊

2.1 Regular pairs for $L^2([0,1])$

For the interval $[a,b]=[0,1]$ let S_n^m be the subspace of polynomials of degree less than m on each subinterval $I_{j,n}=(\frac{j}{2^n},\frac{j+1}{2^n})$, $j=0,1,...,2^n-1$ (for example, S_n^2 is the subspace of polynomials of degree 0 or 1 on each $I_{j,n}$ and S_n^1 is the subspace of piecewise constant functions on each $I_{j,n}$). Then, $\dim(S_n^m)=m2^n$ and $S_0^m\subset S_1^m\subset...\subset S_n^m\subset...\subset\bigcup_{n=0}^{\infty}S_n^m:=S^m$.

Since $\overline{S^m}=L^2([0,1])$, the condition of denseness is satisfied.

We will construct a basis for S_n^m adapting and normalizing the Legendre polynomials of degree $l=0,1,...,m-1$, to each of the subintervals $I_{j,n}$, $j=0,1,...,2^n-1$:

$$S_n^m=spam\{p_l^{j,n},\ j=0,1,...,2^n-1,\ l=0,1,...,m-1\}$$

with

$$p_l^{j,n}(x)=\frac{P_l(2^n(2x-\frac{1+2j}{2^n}))\chi_{I_{j,n}}}{\left\|P_l(2^n(2x-\frac{1+2j}{2^n}))\chi_{I_{j,n}}\right\|},$$

for $P_l(x)$ the Legendre polynomial of degree l on $[-1,1]$ and $\chi_{I_{j,n}}$ the characteristic function of the subinterval $I_{j,n}$.

In order to simplify the notation, we rename $q_l^{i,n}:=p_l^{i,n+1}$ and choose the following subspaces:

$$X_n=S_n^2=span\{p_0^{0,n},p_1^{0,n},p_0^{1,n},p_1^{1,n},...,p_0^{2^n-1,n},p_1^{2^n-1,n}\}$$

$$Y_n=S_{n+1}^1=span\{q_0^{0,n},q_0^{1,n},...,q_0^{2^{n+1}-1,n}\}.$$

Note that $\dim(X_n)=2.2^n=2^{n+1}=1.2^{n+1}=\dim(Y_n)$ and, as we have chosen orthogonal polynomials, condition (7) is also satisfied and uniqueness of the solution of the equation (3) is guaranteed for each n.

Renaming the basis - $\varphi_i^n:=p_0^{\frac{i-1}{2},n}$ for i odd, $\varphi_i^n:=p_1^{\frac{i-2}{2},n}$ for i even and $\psi_j^n:=q_0^{j,n}$ - it is easy to show that $\{X_n,Y_n\}$ is a regular pair, since the correlation matrix $G(X_n,Y_n)=[\langle\ \varphi_i^n,\psi_j^n\ \rangle]=\int_0^1\varphi_i^n(x).\psi_j^n(x)dx$ is a $2^{n+1}\times2^{n+1}$ matrix composed by 2^n identical 2×2 blocks on principal diagonal, $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{-\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}}\end{bmatrix}$, and $G_+(X_n,Y_n):=\frac{1}{2}[G(X_n,Y_n)+G(Y_n,X_n)]$ is, in turn, a $2^{n+1}\times2^{n+1}$ matrix with definite positive blocks on its principal diagonal: $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}-\sqrt{3/2}}{4}\\ \frac{\sqrt{2}-\sqrt{3/2}}{4} & \frac{\sqrt{3}}{2\sqrt{2}}\end{bmatrix}$.

In addition, computations to obtain u_n are easy because test functions are piecewise constant. In the numerical example we observe that a small n is enough to obtain good accuracy.

3 ITERATED PETROV-GALERKIN METHOD

As the equation being solved is of the form $u - Ku = g$, or $u = g + Ku$, an almost natural iteration procedure is to obtain a new approximation u_n^* from u_n by evaluating $u_n^* = g + Ku_n$. This first iteration applied to Galerkin method has been studied since the '70s because, under appropriated conditions on K and g, it reveals an interesting phenomenon called "superconvergence" (see Chandler (1979) and Sloan (1976)), as the order of convergence can be notably improved.

A theorem from Sloan (1990, p. 42) guarantees the existence of a unique solution u_n^* to (6) and the improvement of the order of convergence of the iterated approximation for any projection method. A theorem from Chen and Xu (1998, p. 419) explains the superconvergence in Petrov-Galerkin scheme applied to Fredholm equations of the second kind, under the same conditions of the Theorem 2.1 enunciated in Section 2. Explicitly, the approximation u_n^* satisfy

$$\|u - u_n^*\|_2 \le C.\operatorname*{ess\,sup}_{s\in[a,b]}\Big[\inf_{\psi\in Y_n}\|k(s,.) - \psi\|_2\Big].\inf_{x\in X_n}\|x - u\|_2 \qquad (8)$$

for u the unique solution in $L^2([0,1])$ of equation (1). This result shows that the improvement of the order of convergence by iteration is due to the approximation of the kernel k by members of test subspace, Y_n.

In our case, test subspaces Y_n, are S_{n+1}^1, piecewise constant functions on subintervals $I_{j,n} = (\frac{j}{2^{n+1}}, \frac{j+1}{2^{n+1}})$, $j = 0,1,\dots,2^{n+1} - 1$.

Following Chandler (1979, p. 67), suppose f is a Lipschitz function on the interval I, with Lipschitz constant L, and $\psi_{\frac{1}{2}}$ is the piecewise constant function defined as $\psi_{\frac{1}{2}}(t) = f(t_i + \frac{h}{2})$ for $t \in I_i = [t_i, t_i + h]$, with $I = \bigcup I_i$ a regular partition of I with norm h. Then, $|\, f(t) - \psi_{\frac{1}{2}}(t)\,| \le \frac{1}{2}h.L$ for $t \in I_i, \forall i$ and, consequently, $\|\, f - \psi_{\frac{1}{2}}\,\|_\infty \le \frac{1}{2}h.L$.

If we assume that the kernel $k:[0,1]\times[0,1] \to R$ satisfies that, for each $s \in [0,1]$, $k(s,.) = k_s(.)$ is a Lipschitz function on the interval $[0,1]$ with Lipschitz constant L_s, we can write, for $\psi_{\frac{1}{2}} \in S_{n+1}^1$ defined from k_s, $\|\, k_s - \psi_{\frac{1}{2}}\,\|_\infty \le \frac{1}{2}\frac{1}{2^{n+1}}L_s$ and, then, $\inf_{\psi\in Y_n}\|k_s - \psi\|_\infty \le \frac{1}{2^{n+2}}L_s$.

For each $s \in [0,1]$ it is $\|k_s - \psi\|_2 = (\int_0^1 |k(s,t) - \psi(t)|^2 dt)^{\frac{1}{2}} \le \|k_s - \psi\|_\infty.1$ and $\inf_{\psi\in Y_n}\|k_s - \psi\|_2 \le \frac{1}{2^{n+2}}L_s$ and, consequently, $\operatorname*{ess\,sup}_{s\in[0,1]}\Big[\inf_{\psi\in Y_n}\|k(s,.) - \psi\|_2\Big] \le \frac{1}{2^{n+2}}\operatorname*{ess\,sup}_{s\in[0,1]}[L_s]$.

Finally, from (8),

$$\|u - u_n^*\|_2 \le C.\frac{1}{2^{n+2}}\operatorname*{ess\,sup}_{s\in[0,1]}[L_s].\inf_{x\in X_n}\|x - u\|_2,$$

and we can state the following

Result 3.1: Let the kernel $k:[0,1]\times[0,1] \to R$ in equation (1) satisfy that, for each $s \in [0,1]$, $k(s,.) = k_s(.)$ is a Lipschitz function on the interval $[0,1]$ with Lipschitz constant L_s, and

$\operatorname*{ess\,sup}_{s\in[0,1]}[L_s] < \infty$ *then, the solution* u_n^* *of the iterated Petrov-Galerkin method satisfies* $\|u-u_n^*\|_2 \le C.\frac{1}{2^{n+2}} \operatorname*{ess\,sup}_{s\in[0,1]}[L_s].\inf_{x\in X_n}\|x-u\|_2$ *and the order of convergence is improved.*

Note that, if $\left| \dfrac{\partial k}{\partial t}(s,t) \right| < M < \infty, \forall (s,t) \in [0,1]\times[0,1]$, the condition is fulfilled.

4 NUMERICAL EXAMPLE

In this section we illustrate the performance of the iterated Petrov-Galerkin method using the proposed basis for the regular pair of subspaces of test and trial functions, $X_n = S_n^2$ and $Y_n = S_{n+1}^1$, in a typical example with known solution.
The equation is

$$u(t) - \tfrac{1}{2}\int_0^1 e^{st^2} u(s)ds = e^t + \frac{1-e^{t^2+1}}{2(t^2+1)} \quad \forall t \in [0,1] \tag{9}$$

and the exact solution is $u(t) = e^t$.

The linear operator $K: L^2([0,1]) \to L^2([0,1])$, $K[u(t)] = \frac{1}{2}\int_0^1 e^{st^2} u(s)ds$, is compact and 1 is not an eigenvalue, since $\|K\| \le (\int_0^1\int_0^1 k^2(s,t)dsdt)^{\frac{1}{2}} \le 0.876$ and for a compact linear operator every eigenvalue λ satisfies $|\lambda| \le \|K\|$. Then, the existence and uniqueness of the solution of (9) is guaranteed and the convergence of the Petrov-Galerkin method is assured.

In Figure 1 we plot the exact solution together with the approximations u_0, u_1, u_2 and u_3, and in Figure 2, with u_0^*, u_1^*, u_2^* and u_3^*. In this last case the approximations and the real solution are indistinguishable

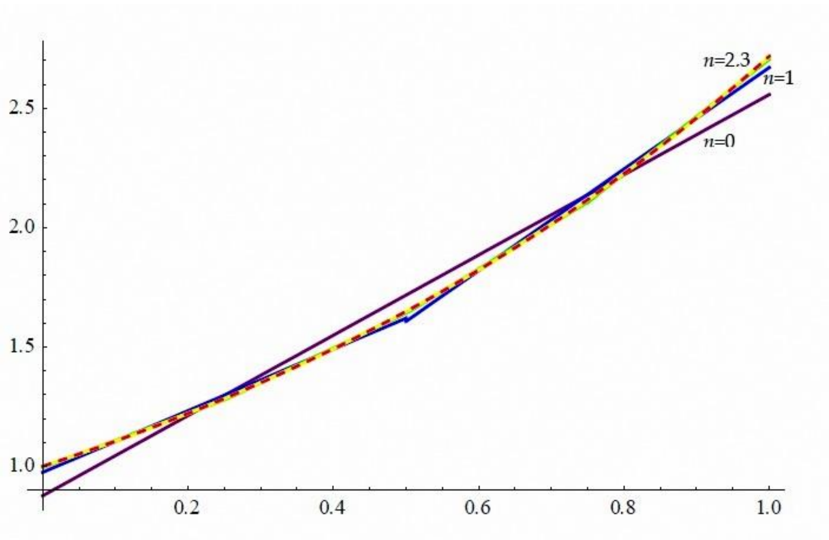

Figure 1: Approximations to the exact solution of equation (9) before iteration. In purple for $n=0$, in blue for $n=1$, in green for $n=2$, in yellow for $n=3$ and, dashed in red, the exact solution.

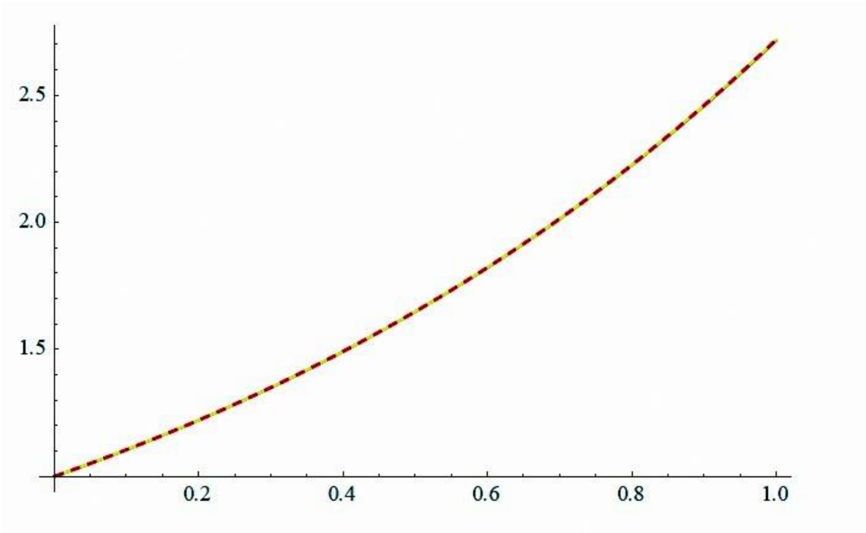

Figure 2: The same approximations after the iteration are graphically indistinguishable from the exact solution of equation (9).

5 CONCLUSIONS AND FUTURE WORK

A well-known method is applied, with quite simple calculations by choosing appropriate subspaces for projecting. Iteration shows to be a very simple way for improving convergence in a remarkable way and better orders of convergence can be shown.

In future works, we intend to explore the performance of iterated Petrov-Galerkin method for Fredholm equations described by operators with kernels with some kind of singularity.

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