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A SUCTION INDUCED MODIFICATION IN ELASTOPLASTIC BEHAVIOUR OF PARTIALLY SATURATED SOILS

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Abstract. The main goal of the present paper is to carry out a plastic mathematical model for partially saturated soils based on the Critical State Theory (J.H. Atkinson and P.L. Bransby, McGraw Hill, (1978)). For this scope, a plastic yield criterion for saturated soil (H.A. Di Rado, et.al. Informacion Tecnologica,10(6):165-175(1999)), was re formulated for partially saturated soils by the inclusion of the matric suction as an independent parameter. Another two adjustment parameters were included and a kinematic and an isotropic expansion of the yield function were combined. The computational implementation was carried out using the Finite Elements Method. Diverse results were presented and some discussions about the pore pressures and the vertical displacements were addressed showing the adaptability of the model to highly compressive non saturated soils.

1 INTRODUCTION

The Civil Engineering presents a lot of problems in which the soil does not behave as an elastic continuum medium, and the theory of plasticity have to be considered. In this situation the soil is treated as an idealized material that behaves elastically until a certain state of stress in which the failure or yielding of the soil is attained. After that, the granular structure behaviour is simulated using constitutive relationships that keep in mind the path dependence of the stresses (Zienkiewicz and Taylor, 1991; Beneyto et al., 2005).

In this paper, an elastoplastic model of saturated soils based on the Critical State Theory (Atkinson and Bransby, 1978; Di Rado et al., 1999)) is re formulated to undertake the solution of the three phase non saturated soil problem by the incorporation of the suction (p^c) as an independent parameter.

2 CLASSICAL PLASTICITY THEORY

Classical plasticity theories introduce a yield surface defined by.

$$F\left(\boldsymbol{\sigma},\kappa\right)=0 \tag{1}$$

where κ is a set of internal variables (hardening parameters) which modify the evolution of the yield surface in the stress space.

The consistency condition requires stresses to remain on the yield surface (1) when yielding occurs. Otherwise, for elastic state, stress points rest inside that surface.

$$F\left(\boldsymbol{\sigma},\kappa\right)<0 \tag{2}$$

The unit outer normal to surface (1)

$$\boldsymbol{n}_f=\frac{\partial F/\partial\boldsymbol{\sigma}}{\left|\partial F/\partial\boldsymbol{\sigma}\right|} \tag{3}$$

characterizes the loading-unloading direction

$$\boldsymbol{n}_f\cdot d\boldsymbol{\sigma}\;\begin{array}{l}<0\;(unloading)\\=0\;(neutral\;loading)\\>0\;(loading)\end{array} \tag{4}$$

and the direction of the plastic flow in the associative plasticity case.

$$d\boldsymbol{\epsilon}^p=\frac{1}{A}\left(\boldsymbol{n}_f\;d\boldsymbol{\sigma}\right)\boldsymbol{n}_f \tag{5}$$

The plastic module, A, is also obtained from consistency rules. For example, for strain hardening, $d\boldsymbol{\kappa}=d\boldsymbol{\kappa}\left(d\boldsymbol{\epsilon}^p\right)$, the condition that the stress point rest on the yield surface can be written as

$$dF=\frac{\partial F}{\partial\boldsymbol{\sigma}}d\boldsymbol{\sigma}+\frac{\partial F}{\partial\boldsymbol{\kappa}}d\boldsymbol{\kappa}=0 \tag{6}$$

This gives, by comparison with equations (3) and (6), the *Hardening module* (Simo and Hughes, 1998) for associative plasticity.

$$A=-\frac{\partial F}{\partial\kappa}\frac{d\kappa}{d\lambda} \tag{7}$$

3 THE CRITICAL STATE MODEL

The yield criterion of the Critical State Theory applied to saturated soils (Atkinson and Bransby, 1978) states a yield surface, $F\left(p^{'},q,\theta\right)=0$, defined in term of the stress invariant, $p^{'}$, q, θ, of the Cauchy effective stress tensor, $\boldsymbol{\sigma}^{'}$, being this:

$$\boldsymbol{\sigma}^{'}=\boldsymbol{\sigma}-\boldsymbol{I}\;\boldsymbol{u}_{w} \tag{8}$$

where I is the identity matrix and u_w is the water pore pressure.

$$p^{'}=-\frac{J_1}{3} \tag{9}$$

$$q=\sqrt{3\;J_2^{'}} \tag{10}$$

$$sen\left(3\theta\right)=-\frac{3\sqrt{3}}{2}\frac{J_3^{'}}{J_2^{'3/2}} \tag{11}$$

where J_1 is the first invariant of (8) and $J_2^{'}$ y $J_3^{'}$ are the second and the third invariants of the deviatoric stress tensor of (8) respectively.

It can be seen in the expression (11) that θ could vary between two limits, i.e., $-30°\leq\theta\leq+30°$.

One of the widely used yield criterion based on the Critical State Theory was proposed by Zienkiewicz et al. (1975) and lightly modified by Di Rado and Awruch (1997). The main characteristic of this yield surface is that, it presents well behaviour when lightly overconsolidated soils are modelled.

$$F\left(p^{'},q,\theta\right)=\frac{p^{'}+a}{2}\left[\left(\frac{q}{p^{'}+a}\right)^{2}\frac{1}{tg^{2}\;\phi}+1\right]-p_{co}=0 \tag{12}$$

were: $(2p_{co}-a)$ is the initial pre consolidation pressure and

$$a=\frac{C}{tg\phi}\;,\;tg\phi=\frac{3sen\phi}{\sqrt{3}\;cos\theta-sen\phi sen\theta}\;,\;C=\frac{3ccos\phi}{\sqrt{3}\;cos\theta-sen\phi sen\theta} \tag{13}$$

The c and ϕ coefficients are the cohesion and the internal friction respectively.

The evolution of yield surface is governed by the variation of volumetric strains with the following expression:

$$p_{co}=p_{co}^{0}e^{\chi\epsilon_v^p} \tag{14}$$

where ϵ_v^p is the volumetric plastic strain and χ is a coefficient given by:

$$\chi=-\beta\frac{1+e_0}{\lambda-k} \tag{15}$$

In the previous expression (15) e_0 is the initial void relation, λ and k are compression and expansion index respectively (Atkinson and Bransby, 1978), determined in oedometer tests and β is an adjustment parameter which value is similar to p_{co}^{0} (Di Rado et al., 1999).

3.1 Constitutive relation for saturated soils

The elastoplastic constitutive relation for saturated soils is defined by the plastic flow vector and the hardening module (7).

In order to settle down the plastic flow vector, the chain rule is used:

$$\boldsymbol{a}^T = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial F}{\partial J_1}\frac{\partial J_1}{\partial \sigma'} + \frac{\partial F}{\partial J_2'^{1/2}}\frac{\partial J_2'^{1/2}}{\partial \sigma'} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial \sigma'} = C_1\boldsymbol{a}_1 + C_2\boldsymbol{a}_2 + C_3\boldsymbol{a}_3 \tag{16}$$

where:

$$\boldsymbol{a}_1^T = \frac{\partial J_1}{\partial \boldsymbol{\sigma}'} \quad , \quad \boldsymbol{a}_2^T = \frac{\partial J_2'^{1/2}}{\partial \boldsymbol{\sigma}'} \quad , \quad \boldsymbol{a}_3^T = \frac{\partial \theta}{\partial \boldsymbol{\sigma}'} \tag{17}$$

$$\begin{aligned} C_1 &= \frac{\partial F}{\partial J_1} = \frac{1}{6}\left\{\left[\frac{q}{(p'+a)\,tg\phi}\right]^2 - 1\right\} \\ C_2 &= \frac{\partial F}{\partial J_2'^{1/2}} = \frac{1}{\sqrt{3}}\frac{p}{(p'+a)\,tg\phi}\left[\frac{3}{tg\phi} + tg\;3\theta\frac{\sqrt{3}sen\theta + cos\theta sen\phi}{sen\phi}\right] \\ C_3 &= \frac{\partial F}{\partial \theta} = \frac{1}{2cos\;3\theta}\frac{q}{(p'+a)\,tg\phi\;J_2'}\left[\frac{\sqrt{3}sen\theta + cos\theta sen\phi}{sen\phi}\right] \end{aligned} \tag{18}$$

The relationship between the principal stresses and the stress invariants is given by (Owen and Hinton, 1980):

$$\left\{\begin{array}{c}\sigma_1' \\ \sigma_2' \\ \sigma_3'\end{array}\right\} = \frac{2}{\sqrt{3}}J^{1/2}\left\{\begin{array}{c} sen\left(\theta + \frac{2\pi}{3}\right) \\ sen\theta \\ sen\left(\theta + \frac{4\pi}{3}\right)\end{array}\right\} + \frac{J_1}{3}\left\{\begin{array}{c}1\\1\\1\end{array}\right\} \tag{19}$$

The main advantage of expressing the yield surface in terms of alternative stress invariants, p', q y θ, is that it allows to develop the computer code of the yield function and the flow rule in a general form, by only the specification of three (C_1, C_2 and C_3, (16)) constants for any individual criterion (Viladkar et al., 1995). The equation (7) is directly used to determine the hardening parameter keeping in mind that $d\kappa = d\epsilon_v^p$ for this yielding criterion (strain hardening).

From (14), it can be obtained:

$$ln\frac{p_{co}}{p_{co}^0} = \chi\epsilon_v^p \tag{20}$$

By differentiating (20), the following relation between p_{co} and ϵ_v^p is obtained.

$$\frac{dp_{co}}{p_{co}^0} = \chi d\epsilon_v^p \tag{21}$$

Then, by use of (7) the hardening module can immediately be rewritten as

$$A = \frac{p_{co}}{2}\beta\frac{1+e_0}{\lambda - k}\left[1 - \left(\frac{q}{(p'+a)\,tg\phi}\right)^2\right] \tag{22}$$

4 MODEL ADAPTATION TO PARTIALLY SATURATED SOILS

In the previous section, a mathematical framework for elastoplastic behaviour of saturated soil was presented. The three phase problem of partially saturated soils was reviewed by various authors (Li et al., 1999; Bolzon et al., 1996; Lewis and Schrefler, 1987; Beneyto et al., 2005) and they agreed in consider the matric suction as a decisive variable in the problem. The matric suction is defined by:

$$\boldsymbol{p}^c = \boldsymbol{u}_a - \boldsymbol{u}_w \tag{23}$$

where $\boldsymbol{u}_w$ is the water pore pressure and $\boldsymbol{u}_w$ is the air pore pressure. This paper is focused on the modification of the above mentioned formulation in order to include the matric suction in the mathematical description of the non saturated soil behaviour. Therefore, to carry out this purpose, two different ways to consider the suction is regarded:

1. As another stress function variable.
2. As another hardening parameter.

Nevertheless, in both cases an immediate problem must be overcome. The Critical State Theory states that, on the Critical State Line (C.S.L.) no plastic volumetric strain is allowed (Atkinson and Bransby, 1978; Di Rado, 1997). This is tantamount to say that plastic strain due to suction are tolerated whenever the stress state does not lay on the C.S.L. To fulfil this condition, two guidelines are addressed:

- If the first option is used, another plastic potential function has to be considered. This particular function must be suction independent. The problem is that the non associate plasticity leads to non symmetrical formulations due to the lost of symmetry of the elasto-plastic matrix (Khalili and Loret, 2001).
- If the second option is selected, the plastic potential function and the yield function match exactly, thus, no lost of symmetry occur. Although, the plastic multiplier must be suction independent.

It is clear that the fact of choosing the second option is somewhat contradictory as will be seen later on. However this assumption, from a theoretical viewpoint, is not less accurate than the case with non associated plasticity regarding that, for that case, the principle of maximum plastic dissipation (Simo and Hughes, 1998) is not satisfied. Furthermore, the second alternative meets the Critical State Theory requirement previously pointed out.

Therefore, the second option was adopted with two different and simultaneous expansion of the yield surface, both related to the suction changes, p^c, with respect to the initial suction value, p^c_0.

4.1 Kinematic expansion

The kinematic expansion of the surface is carried out modifying the coefficient a in (12), in order to turn it a function of the current increment of the suction. To get the first purpose, a new coefficient, k, is introduced. Thus a remains:

$$a = \frac{c}{tg\phi} + |p^c - p^c_0|\ k \tag{24}$$

When the suction increases, both the C.S.L. and the yield surface are dragged to the left in the p', q diagram (see picture 1).

On the other hand, a decrease in the suction value moves the surface towards the position that would be occupied by a surface standing for the same kind of soil but in saturated condition, which is accurate from a physical viewpoint and agree with Li et al. (1999). However, experimental results (Fredlund et al., 1978) indicate that a growing suction should cause the surface to grow as well. A kinematic hardening is unable to reproduce this situation; hence, an isotropic hardening is required.

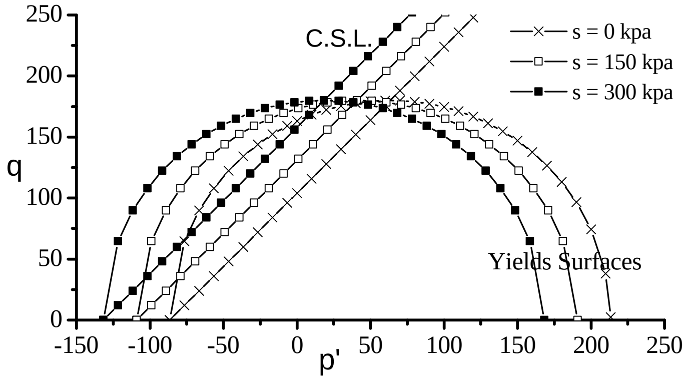

Figure 1: Kinematic expansion

4.2 Isotropic expansion

To improve the behaviour of the elastoplastic model, accordingly to what was suggested in the previous paragraph; a modification in the term p_{co} of (12) was adopted and is given by

$$p_{co} = \left(p^0_{co} + H_w\right) e^{\left(\chi\, \epsilon^p_v\right)} \tag{25}$$

where H_w, has a linear variation with the suction (Alonso et al., 1999) depending on the regional soil characteristics (Fredlund and N.R., 1977).

$$H_w = m\ \left|p^c - p^c_0\right| \tag{26}$$

The change in p_{co} cause module A to be modified. The procedure for the determination of the new hardening module is similar to that used in (22) but keeping in mind two hardening parameters, $\kappa_1 = \epsilon^p_v$ and $\kappa_2 = p^c$.

$$A = -\frac{\partial F}{\partial \kappa_1}\frac{d\kappa_1}{d\lambda} - \frac{\partial F}{\partial \kappa_2}\frac{d\kappa_2}{d\lambda} \tag{27}$$

But according to what was assumed previously,

$$\frac{d\kappa_2}{d\lambda} = \frac{dp^c}{d\lambda} = 0 \tag{28}$$

Now the new hardening module will be:

$$A = \frac{p_{co} + H_w}{2}\beta\frac{1+e_0}{\lambda - k}\left[1 - \left(\frac{q}{(p' + a)\, tg\phi}\right)^2\right] \tag{29}$$

Therefore, neither volumetric expansion nor hardening process will occur when the C.S.L. is reached, because the plastic multiplier (29) becomes null. Picture 2 shows the surface expansion with suction increment, considering the modifications imposed by (24), (25) y (29). A three-dimensional graphic representation of the complete model can be seen in figure 3.

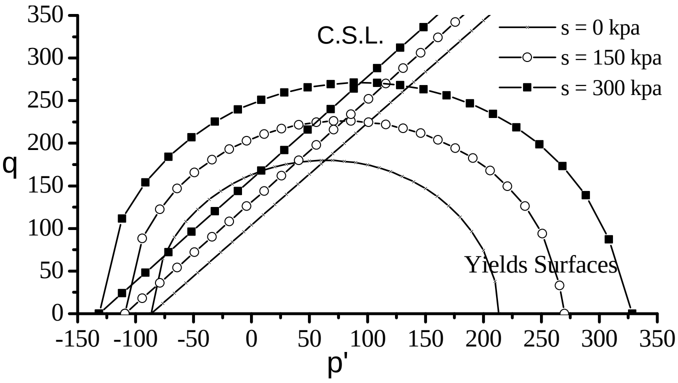

Figure 2: Both expansion

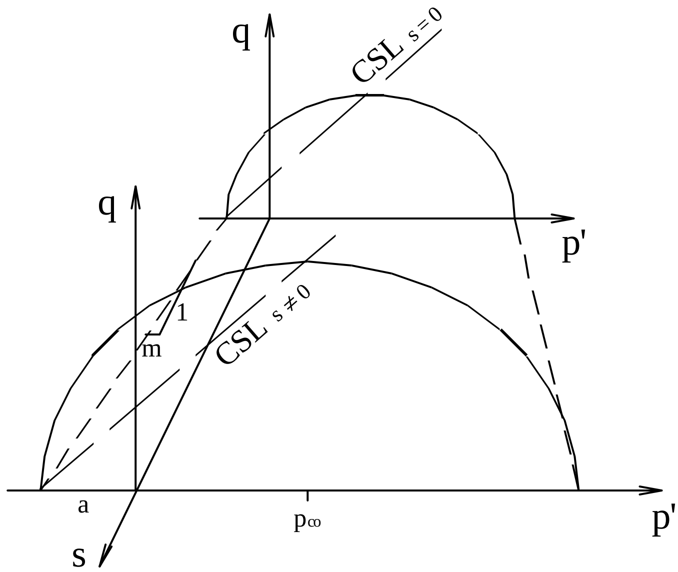

Figure 3: Yield surface

5 NUMERICAL EXAMPLES

5.1 A soil column problem

The following example consists on a stratum of infinite longitude and $5m$ of depth, of partially saturated soil loaded with $q = 100kpa$. The soil properties are: Young module $E = 1500kpa$, Poison coefficient $\nu = 0,3$, Vertical permeability $k_y = 8,64 * 10^{-5}m/dia$, cohesion $c = 50kpa$, internal friction $\phi = 30°$, initial saturation degree $S_r = 0,85$, grains compressibility $k_s = 1 * 10^6kpa$. The adjustment coefficients corresponding to the proposed plastic model are $m = 0,5$, $k = 1,5$ and the initial preconsolidation is $p_{co} = 100kpa$.

For the finite element mesh, serendipity elements of 20 nodes for displacement and 8 nodes for pores pressures (water and air pressures) were used (see figure 4). It was allowed the vertical

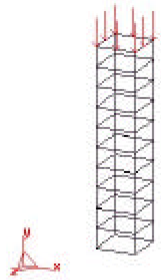

Figure 4: Finite element mesh

displacement of lateral nodes meanwhile all displacement were restricted for the bottom nodes. Both, pore pressures at the top surface, were set to zero.

In figure 5 the evolution of the water pore pressure with time is presented. The mathematical model must meet the principle of minimum potential energy condition and this fact provokes the water to support an increasing load due to the relative lost of the soil structure stiffness because of the plasticity effect.

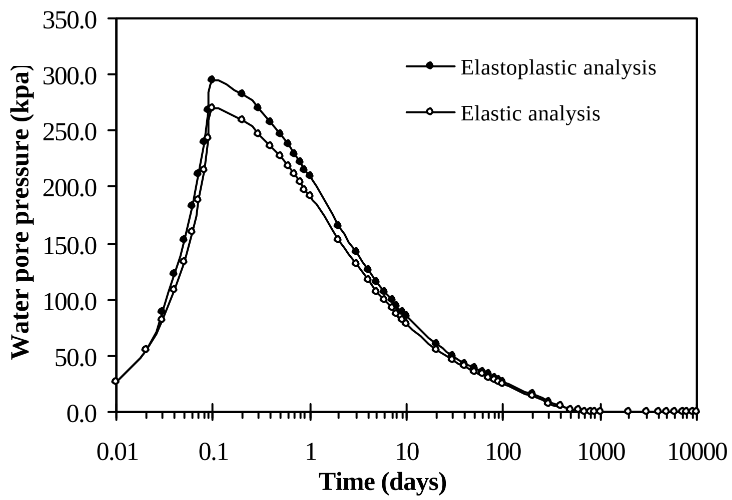

Figure 5: Evolution of pore pressure

There is another important issue to be revised for this example: The vertical displacement changes due to the modifications introduced to the original plastic yield criterion presented in paragraphs 4.1 and 4.2.

The figure 6 shows a displacement vs. time plot obtained varying the plastic yield criterion according to four situations: 1) Using merely kinematic hardening. 2) Using only isotropic hardening. 3) Using a combination of the previous. 4) Using the model proposed for saturated soils.

In the case (1) the suction pushes the yield surface towards the left (in the direction of negative effective mean pressure, see figure 1), whereas the stress path, for this load process, is bended to de right (in the $p' - q$ plane). Thus, the plastic effect is of great magnitude and this fact may be verified considering that the vertical displacements are maximized.

The cases (2) and (4) are essentially the same. For both cases the yield surface grows with the plastic advance (see figure 2.) almost in the same way with only one difference: in the case (2) the suction speeds up the growing process and therefore a minor vertical displacement than

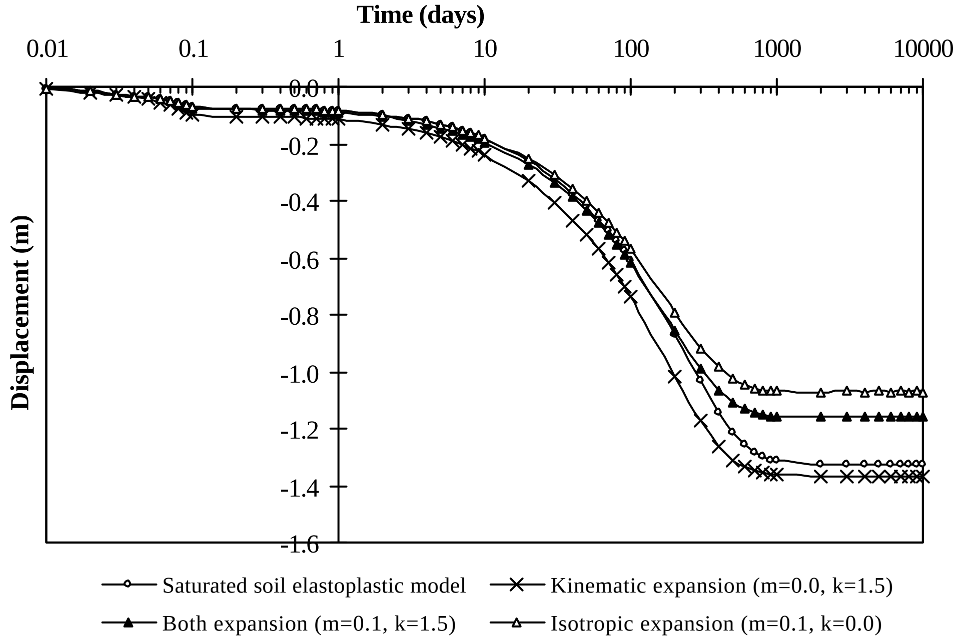

Figure 6: Displacement vs. time for different plastics parameters of the yield criterion

the case (4) was obtained. For the case (3), the kinematic hardening effect is balanced with the isotropic one resulting in an intermediate vertical displacement. This kind of compensation was pointed out in reference (Bolzon et al., 1996)

5.2 Isolated footing

The following example considers the elastoplastic problem of a flexible isolated footing. The resisting portion of unsaturated soil is a cube with $5m$ deep. The initial saturated degree is 85% for the whole mass. The load transmitted by the footing was set to $q = 100kpa$. All the soil properties are the same of example 5.1. Only the initial preconsolidation pressure was changed to $p_{co} = 600kpa$. The additionally horizontal permeability was set to $k_y = 8,64 * 10^{-4}m/dia$. Furthermore, due to the double symmetry of the problem only one quarter of the whole mass of soil was modelled (see picture 9).

In the figure 7 the dissipation of the water pore pressure on a node located $0,5m$ of depth is represented, while in the picture 8, the vertical displacement along the time of the same point is plotted. To conclude, figure 9 shows the water pore pressures levels on the second day.

6 CONCLUSIONS

A plastic yield function based on critical state theory for partially saturated soil was presented. The effective cohesion is apparently augmented by the inclusion of the influence of matric suction according to what was suggested in reference (Li et al., 1999). The recommendations addressed in reference (Fredlund et al., 1978) were also considered through an isotropic hardening with hardening parameters proportional to matric suction. The obtained outcomes for both soil column and isolated function reflect the yield function properties previously outlined.

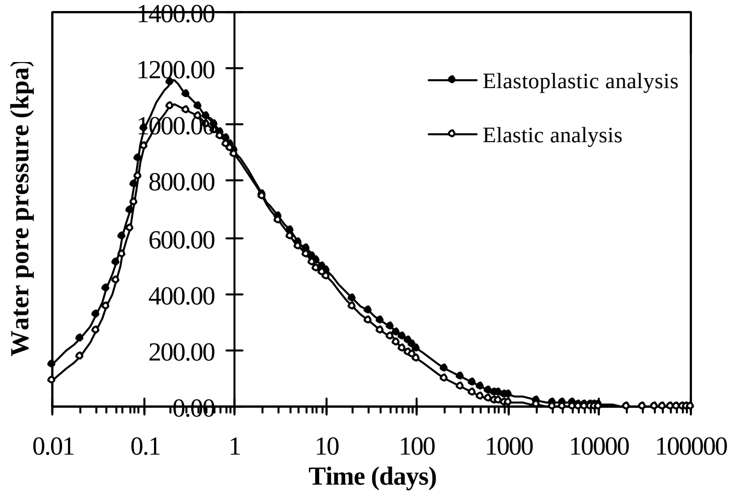

Figure 7: Disipation of water pore pressure

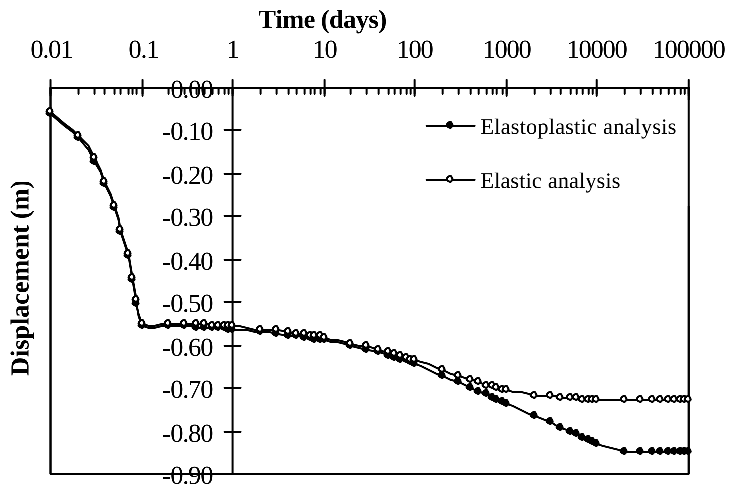

Figure 8: Displacement vs. Time

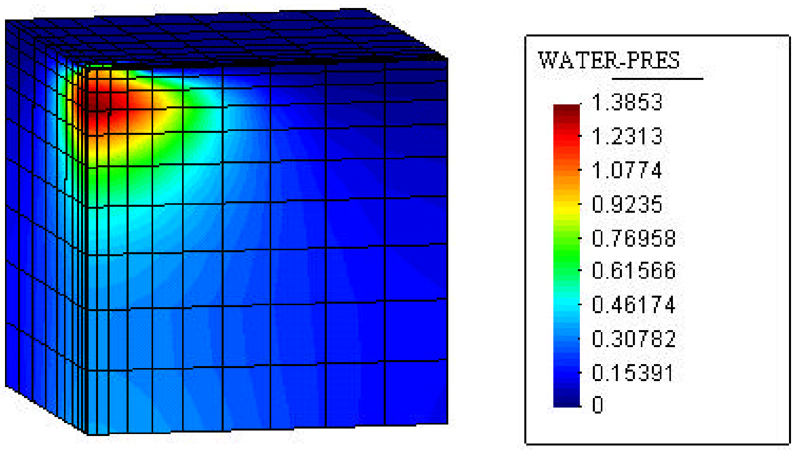

Figure 9: State of the water pore pressures to the second day and finite element mesh

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