

## A REGULARIZATION OPERATOR FOR THE SOURCE APPROXIMATION OF A TRANSPORT EQUATION

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**Abstract.** Source identification problems have multiple applications in engineering such as the identification of fissures in materials, determination of sources in electromagnetic fields or geophysical applications, detection of contaminant sources, among others. In this work we are concerned with the determination of a time-dependent source in a transport equation from noisy data measured at a fixed position. By means of Fourier techniques can be shown that the problem is ill-posed in the sense that the solution exists but it does not vary continuously with the data. A number of different techniques were developed by other authors to approximate the solution. In this work, we consider a family of parametric regularization operators to deal with the ill-posedness of the problem. We proposed a way to select the regularization parameter as a function of noise level in data in order to obtain a regularized solution that approximate the unknown source. We find a Hölder type bound for the error of the approximated source when the unknown function is considered to be bounded in a given norm. Numerical examples illustrate the convergence and stability of the method.

## 1 INTRODUCTION

Source identification problems are inverse problems of great interest due to the amount of applications in different disciplines. For instance, in the problems of source identification we can find applications in heat conduction processes (Hansen and O’Leary, 1993), electromagnetic theory (El Badia and Ha-Duong, 2000), contaminant detection (Li et al., 2006) and detection of tumor cells (Macleod, 1999). The determination of the foci of pollution of the layers of groundwater triggers environmental and health problems for the population. The identification of sources of pollution in groundwater can be modeled by a transport equation where its input represents the mean concentration of contaminants per unit average of effective porosity (Li et al., 2006; Sun, 1996). As a first approach, this work focus on the determination of a source for a one-dimensional equation based on noisy measurements taken in a fixed position, in an unbounded domain. This is an ill-posed problem in the sense of Hadamard (Hadamard, 1923) because its solution does not depend continuously on data. In particular, the high frequency components in arbitrarily small data errors can lead to arbitrarily large errors in the result. For this type of problems, regularization methods are widely used in the literature in order to obtain a stabilized approximate solution (Fu, 2004; Johansson and Lescnic, 2008). Here, we designed a (parametric) family of regularization operators (see (Engel et al., 1996)) that compensates the factor that causes the instability of the inverse operator. These operators approximate the given ill-posed problem by a family well-posed problems. To this end, an *a priori* or *a-posteriori* rule (*parameter choice rule*) must be selected to choose the regularization parameter. For more details see (Engel et al., 1996), (Kirsch, 2011). The proposed family of regularization operators provides a framework in the theory of operators for the modified regularization method used, for instance, in (Qian et al., 2006, 2007; Xiao et al., 2012; Z. Zhao et al., 2014). Assuming that the source is bounded in a given Hilbert space  $H^p$ , we propose a parameter choice rule that depends on  $p$  and on the data noise level. We demonstrate the stability and convergence of the regularization family and obtain a Hölder type bound for the estimation error. Numerical examples illustrate its performance including the calculation of the error and the theoretical bound.

## 2 THE PROBLEM OF THE SOURCE DETERMINATION

We focus on the problem of determining the source  $f$  for the one-dimensional transport equation from noisy data measurements with conditions. Specifically, we look for the source  $f$  that satisfies the system

$$\left\{ \begin{array}{ll} u_t(x, t) = \alpha^2 u_{xx}(x, t) - \beta u_x(x, t) - \nu u(x, t) + f(t), & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = 0, & x \in \mathbb{R} \\ u(0, t) = 0, & t > 0 \\ u(x_0, t) = y(t), & t > 0, \quad x_0 > 0 \\ \lim_{x \rightarrow \infty} |u(x, t)| < M & t > 0, \text{ for some } M \in \mathbb{R} \end{array} \right. \quad (1)$$

where  $\alpha^2, \nu > 0$ ,  $\beta \geq 0$ . In addition, we assume that  $u(x, \cdot), f(\cdot) \in L^2(\mathbb{R})$  are unknown functions and that  $y \in L^2(\mathbb{R})$  can be measured with certain noise level  $\delta$ , i.e., the data function  $y_\delta \in L^2(\mathbb{R})$  satisfies  $\|y - y_\delta\|_{L^2(\mathbb{R})} \leq \delta$ .

We define all functions to be zero for  $t < 0$ . By means of the Fourier transform, the problem

can be written in the frequency space as

$$\begin{cases} -\alpha^2 \hat{u}_{xx}(x, \xi) + \beta \hat{u}_x(x, \xi) + z(\xi) \hat{u}(x, \xi) = \hat{f}(\xi), & \xi, x \in \mathbb{R}, \\ \hat{u}(0, \xi) = 0, & \xi \in \mathbb{R} \\ \hat{u}(x_0, \xi) = \hat{y}(\xi), & \xi \in \mathbb{R}, x_0 > 0 \\ \lim_{x \rightarrow \infty} |\hat{u}(x, \xi)| < M^* & \xi \in \mathbb{R}, \text{ for some } M^* \in \mathbb{R}, \end{cases} \quad (2)$$

where  $z(\xi) = \nu + i\xi \in \mathbb{C}$ .

Solving the second order linear ordinary differential equation (2) we obtain

$$\hat{f}(\xi) = \Lambda(\xi) \hat{y}(\xi) \quad \text{where} \quad \Lambda(\xi) = \frac{z(\xi)}{1 - e^{\frac{\beta - \sqrt{\beta^2 + 4\alpha^2 z(\xi)}}{2\alpha^2} x_0}}. \quad (3)$$

We introduce the operator  $T$  for  $y \in L^2(\mathbb{R})$ ,

$$Ty = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} \Lambda(\xi) \hat{y}(\xi) d\xi. \quad (4)$$

Note that

$$f(t) = Ty(t). \quad (5)$$

For simplicity, we drop the variable  $t$  in (5) and denote  $f_\delta = Ty_\delta$ . Hence

$$f - f_\delta = T(y - y_\delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} \Lambda(\xi) (\hat{y}(\xi) - \hat{y}_\delta(\xi)) d\xi. \quad (6)$$

We observe that the factor  $\Lambda(\xi)$  increases without bound as  $|\xi| \rightarrow \infty$  amplifying the high frequency components of the error  $\hat{y}(\xi) - \hat{y}_\delta(\xi)$  which is an undesirable behavior (Hadamard, 1923).

Regularization methods are commonly used in dealing with unstable solutions. Regarding the determination of a source for a parabolic equation, most of the research articles that use certain regularization techniques are restricted to particular cases. In (Sivergina et al., 2003) the author focus on a convection-diffusion equation, while in (Dou and Fu, 2009), (Dou et al., 2009), (Trong et al., 2005), (Trong et al., 2006), (Yang and Fu, 2010), (Yan et al., 2010), (Z. Zhao et al., 2014) only diffusion is considered. In this work we focus on a general one dimensional advection-diffusion transport equation with a time dependent source.

We define a family of operators and an *a-priori* parameter choice rule (Engel et al., 1996), (Kirsch, 2011) in order to regularize the solution to the inverse source problem (5). The stability and convergence of the proposed regularization family is analyzed and an error bound is obtained based on the data noise level, assuming some smoothness on the source. The performance of this approach is numerically illustrated. A comparison with the unregularized solution is included.

### 3 INVERSE PROBLEM REGULARIZATION

**Definition (Kirsch, 2011)** Let  $T : Y \rightarrow X$ ,  $X$  and  $Y$  be Hilbert spaces and  $T$  an unbounded operator. A regularization strategy for  $T$  is a family of linear and bounded operators

$$R_\mu := Y \rightarrow X, \quad \mu > 0, \quad / \quad \lim_{\mu \rightarrow 0} R_\mu y = Ty, \quad \forall y \in Y. \quad (7)$$

Let us define the parametric family of integral operators  $R_\mu : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  for  $\mu \in (0, 1)$ ,

$$R_\mu y := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} \frac{\Lambda(\xi)}{1 + \mu^2 \xi^2} \hat{y}(\xi) d\xi \tag{8}$$

where  $\Lambda(\xi)$  given in (3) and the denominator  $1 + \mu^2 \xi^2$  is introduced for stabilization purposes.

**Theorem 3.1** *Let us consider the problem of identifying  $f$  from noisy data  $y_\delta(t)$  measured at a given position  $x_0 > 0$ , where  $\delta$  is the data noise level and  $u$  and  $f$  satisfy*

$$\begin{cases} u_t(x, t) = \alpha^2 u_{xx}(x, t) - \beta u_x(x, t) - \nu u(x, t) + f(t), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = 0, & x \in \mathbb{R} \\ u(0, t) = 0, & t > 0 \\ \lim_{x \rightarrow \infty} |u(x, t)| < M & t > 0, \text{ for some } M \in \mathbb{R}. \end{cases} \tag{9}$$

Let  $\{R_\mu\}$  be the family of operators defined by (8). Then, for every  $y(t) = u(x_0, t)$  there exists an a-priori parameter choice rule for  $\mu > 0$  such that the pair  $(R_\mu, \mu)$  is a convergent regularization method for solving the identification problem (5).

**Proof** The factor  $\frac{\Lambda(\xi)}{1 + \mu^2 \xi^2}$  is bounded for all  $\xi$  since it is continuous for all  $\xi \in \mathbb{R}$  and

$$\lim_{\xi \rightarrow \pm\infty} \left| \frac{\Lambda(\xi)}{1 + \mu^2 \xi^2} \right| = \lim_{\xi \rightarrow \pm\infty} \left| \frac{z(\xi)}{\left(1 - e^{\frac{\beta - \sqrt{\beta^2 + 4\alpha^2 z(\xi)}}{2\alpha^2} x_0}\right) (1 + \mu^2 \xi^2)} \right| < \infty.$$

Hence, for all  $\mu > 0$ ,  $R_\mu$  is a continuous operator and  $R_\mu \rightarrow T$  pointwise on  $L^2(\mathbb{R})$  as  $\mu \rightarrow 0$  for  $T$  given in (4). Therefore, by Proposition 3.4 in (Engel et al., 1996),  $R_\mu$  is a regularization for  $T$  and for every  $y(t) = u(x_0, t)$  there exists an a-priori parameter choice rule  $\mu$  such that  $(R_\mu, \mu)$  is a convergent regularization strategy for solving (5). The regularized solution is given by

$$f_{\delta, \mu} = R_\mu y_\delta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} \frac{\Lambda(\xi)}{1 + \mu^2 \xi^2} \hat{y}_\delta(\xi) d\xi. \tag{10}$$

**Remark** We observe that the proposed family of regularized operators (10) is equivalent to the regularization method proposed in (Yang and Fu, 2011) when taking  $\alpha^2 = 1$ ;  $\beta = \nu = 0$  and  $x_0 = 1$ .

#### 4 ERROR ANALYSIS

In this section we are concerned with the stability and error analysis of the regularization method. We assume that the source  $f$  is bounded in the Sobolev space  $H^p(\mathbb{R})$ ,  $p > 0$ , i.e., for some  $C > 0$  it holds

$$\|f\|_{H^p(\mathbb{R})} := \left( \int_{-\infty}^{\infty} |\hat{f}|^2 (1 + \xi^2)^p d\xi \right)^{1/2} \leq C. \tag{11}$$

Optimal order strategies require a parameter choice rule that depends on the a-priori bound  $C$  in (11). Since in practice this bound is unknown, we look for a regularization parameter that

leads to a convergent strategy that only depends on the noise level of the data  $\delta$ . Some results are now introduced that will be used later to obtain a bound for the regularization error, that is, the error between the source  $f(x)$  and its estimate  $f_{\delta,\mu}(x)$ .

**Lemma 4.1** For  $\omega \in \mathbb{C}$  with  $Re(\omega) > 0$  holds

$$\left| \frac{1}{1 - e^{-\omega}} \right| \leq \frac{1}{1 - e^{-Re(\omega)}}, \quad Re(\sqrt{\omega}) = \sqrt{\frac{Re(\omega)+|\omega|}{2}} \geq \sqrt{Re(\omega)}.$$

**Lemma 4.2** If  $0 < \mu < 1$  then  $\frac{|x|}{1 + x^2\mu^2} \leq \frac{1}{2\mu}, \quad \forall x \in \mathbb{R}$ .

**Lemma 4.3** The function  $g : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  given by  $g(x) = \begin{cases} \frac{x}{1 - e^{-x}} & 0 < x < 1 \\ \frac{1}{1 - e^{-x}} & 1 \leq x \end{cases}$  satisfies

$$g(x) \leq 2, \quad \forall x > 0.$$

**Lemma 4.4** Let  $\alpha^2, \nu, x_0 > 0, \beta \geq 0$  and  $0 < \mu < 1$  then,

$$\left| \frac{\Lambda(\xi)}{1 + \xi^2\mu^2} \right| \leq \frac{1}{\mu^2} \left( \frac{2\alpha^2(2\nu + 1)}{(-\beta + \sqrt{\beta^2 + 4\alpha^2\nu}) x_0} \right).$$

**Proof** From equation (3), Lemma 4.1 and the triangular inequality we have

$$\left| \frac{\Lambda(\xi)}{1 + \xi^2\mu^2} \right| \leq \frac{\nu + |\xi|}{\left(1 - e^{-\left(\frac{-\beta + \sqrt{\beta^2 + 4\alpha^2\nu}}{2\alpha^2}\right) x_0}\right) (1 + \xi^2\mu^2)}. \tag{12}$$

Let us denote  $m = m(\alpha, \beta, \nu) = \frac{-\beta + \sqrt{\beta^2 + 4\alpha^2\nu}}{2\alpha^2}$  and consider two cases:  $m x_0 \geq 1$  and  $m x_0 \in (0, 1)$

Case  $m x_0 \geq 1$ : Using Lemmas 4.2-4.3, for  $0 < \mu < 1$  results

$$\frac{\nu + |\xi|}{(1 - e^{-m x_0}) (1 + \xi^2\mu^2)} \leq 2 \left( \frac{\nu}{1 + \xi^2\mu^2} + \frac{|\xi|}{1 + \xi^2\mu^2} \right) \leq 2\nu + \frac{1}{\mu}. \tag{13}$$

Case  $m x_0 \in (0, 1)$ :

Observe that multiplying and dividing by  $m x_0$  for  $0 < \mu < 1$ , Lemmas 4.2-4.3 imply

$$\begin{aligned} \frac{\nu + |\xi|}{(1 - e^{-m x_0}) (1 + \xi^2\mu^2)} &\leq \frac{2}{m x_0} \left( \frac{\nu + |\xi|}{1 + \xi^2\mu^2} \right) \\ &\leq \frac{2}{m x_0} \left( \frac{\nu}{1 + \xi^2\mu^2} + \frac{1}{2\mu} \right) \leq \left( 2\nu + \frac{1}{\mu} \right) \frac{1}{m x_0}. \end{aligned} \tag{14}$$

Note that for  $0 < \mu < 1$ , equations (13) - (14) yield

$$\left| \frac{\Lambda(\xi)}{1 + \xi^2\mu^2} \right| \leq \frac{1}{\mu^2} \left( \frac{2\alpha^2(2\nu + 1)}{(-\beta + \sqrt{\beta^2 + 4\alpha^2\nu}) x_0} \right).$$

and the proof is completed.

**Theorem 4.5** Consider the inverse problem of determining the source  $f(t)$  in (1). Let  $f_{\delta,\mu}(t)$  be the regularization solution given in (10) and assume that  $\|f\|_{H^p(\mathbb{R})}$  is bounded in  $H^p(\mathbb{R})$  (11). Then choosing the regularization parameter  $\mu^2 = \delta^{\frac{2}{p+2}}$ , there exists a constant  $K$  independent of  $\delta$  such that

$$\|f - f_{\delta,\mu}\|_{L^2(\mathbb{R})} \leq K \max \left\{ \delta^{\frac{p}{p+2}}, \delta^{\frac{2}{p+2}} \right\}. \quad (15)$$

**Proof** From now on, let us denote  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ . Defining  $\hat{f}_\mu(\xi) := \frac{\Lambda(\xi)}{1 + \mu^2 \xi^2} \hat{y}(\xi)$ , one has

$$\begin{aligned} \left| \hat{f}(\xi) - \hat{f}_\mu(\xi) \right| &= \left| \hat{f}(\xi) - \frac{\Lambda(\xi)}{1 + \mu^2 \xi^2} \hat{y}(\xi) \right| = \left| \hat{f}(\xi) \frac{(1 + \xi^2)^{\frac{p}{2}}}{(1 + \xi^2)^{\frac{p}{2}}} \left( 1 - \frac{1}{1 + \xi^2 \mu^2} \right) \right| \\ &\leq \sup_{\xi \in \mathbb{R}} \left| (1 + \xi^2)^{-\frac{p}{2}} \left( 1 - \frac{1}{1 + \xi^2 \mu^2} \right) \right| \left| \hat{f}(\xi) (1 + \xi^2)^{\frac{p}{2}} \right|. \end{aligned} \quad (16)$$

By (10) and the triangle inequality we have that

$$\|\hat{f} - \hat{f}_{\delta,\mu}\| \leq \|\hat{f} - \hat{f}_\mu\| + \|\hat{f}_\mu - \hat{f}_{\delta,\mu}\|. \quad (17)$$

Now, (16)- (17), the definition of  $H^p(\mathbb{R})$ -norm given in (11) and (10) lead to

$$\|\hat{f} - \hat{f}_{\delta,\mu}\| \leq \sup_{\xi \in \mathbb{R}} \left| (1 + \xi^2)^{-\frac{p}{2}} \left( 1 - \frac{1}{1 + \xi^2 \mu^2} \right) \right| \|f\|_{H^p(\mathbb{R})} + \sup_{\xi \in \mathbb{R}} \left| \frac{\Lambda(\xi)}{1 + \xi^2 \mu^2} \right| \|\hat{y} - \hat{y}_\delta\|.$$

From (Yang and Fu, 2010), it holds

$$\sup_{\xi \in \mathbb{R}} \left| (1 + \xi^2)^{-\frac{p}{2}} \left( 1 - \frac{1}{1 + \xi^2 \mu^2} \right) \right| \leq \max \{ \mu^p, \mu^2 \}.$$

Thus, Lemma 4.4 and the assumption  $\|\hat{y} - \hat{y}_\delta\| \leq \delta$  yields to

$$\|\hat{f} - \hat{f}_{\delta,\mu}\| \leq \max \{ \mu^p, \mu^2 \} \|f\|_{H^p(\mathbb{R})} + \frac{\delta}{\mu^2} \left( \frac{2\alpha^2(2\nu + 1)}{(-\beta + \sqrt{\beta^2 + 4\alpha^2\nu}) x_0} \right).$$

By Parseval's identity, the linearity of the Fourier transform and (11), choosing  $\mu^2 = \delta^{\frac{2}{p+2}}$  we obtain

$$\|f - R_\mu y_\delta\| = \|\hat{f} - \hat{f}_{\delta,\mu}\| \leq K \max \left\{ \delta^{\frac{p}{p+2}}, \delta^{\frac{2}{p+2}} \right\}. \quad (18)$$

where  $K = C + \left( \frac{2\alpha^2(2\nu+1)}{(-\beta+\sqrt{\beta^2+4\alpha^2\nu}) x_0} \right)$  and  $C$  is the bound in(11).

## 5 NUMERICAL EXAMPLES

We consider functions  $u$  and  $f$  that satisfy a given transport equations on an interval  $(0, T_M]$  and define a uniform partition  $\mathcal{P} = \{t_j, j = 0, \dots, n\}$  on that interval. A set of numerically simulated noisy data  $\{u(x_0, t_j) = y_\delta = y(t_j) + e_j, t_j \in \mathcal{P}, j = 0, \dots, n\}$  is obtained where  $\{e_j, j = 1, \dots, n\}$  are realizations of normally distributed random variables with mean 0 and standard deviation  $\epsilon < 1$  and  $\delta$  satisfies  $\|y_\delta(t) - y(t)\|_{L^2(\mathbb{R})} \leq \delta$ . Then, we set the regularization parameter  $\mu = \delta^{\frac{2}{p+2}}$  and calculate the approximated solution  $f_{\delta,\mu}(t)$  given by (10).

**Example 5.1** Consider the inverse source problem defined in (1) with modeling parameter values  $\alpha^2 = 0.01, \beta = 0.5, \nu = 1.51$ . Hence, the problem is to determine  $f$  in

$$\begin{cases} u_t(x, t) = 0.01u_{xx}(x, t) - 0.5u_x(x, t) - 1.51u(x, t) + f(t), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = 0, & x \in \mathbb{R}, \\ u(0, t) = 0, & t > 0 \\ u(x_0, t) = y(t), & t > 0. \end{cases}$$

We consider the source  $f$  given by

$$f(t) = \begin{cases} 6.51e^{-t}, & 20 > t > 0, \\ 0, & t \geq 20. \end{cases} \quad (19)$$

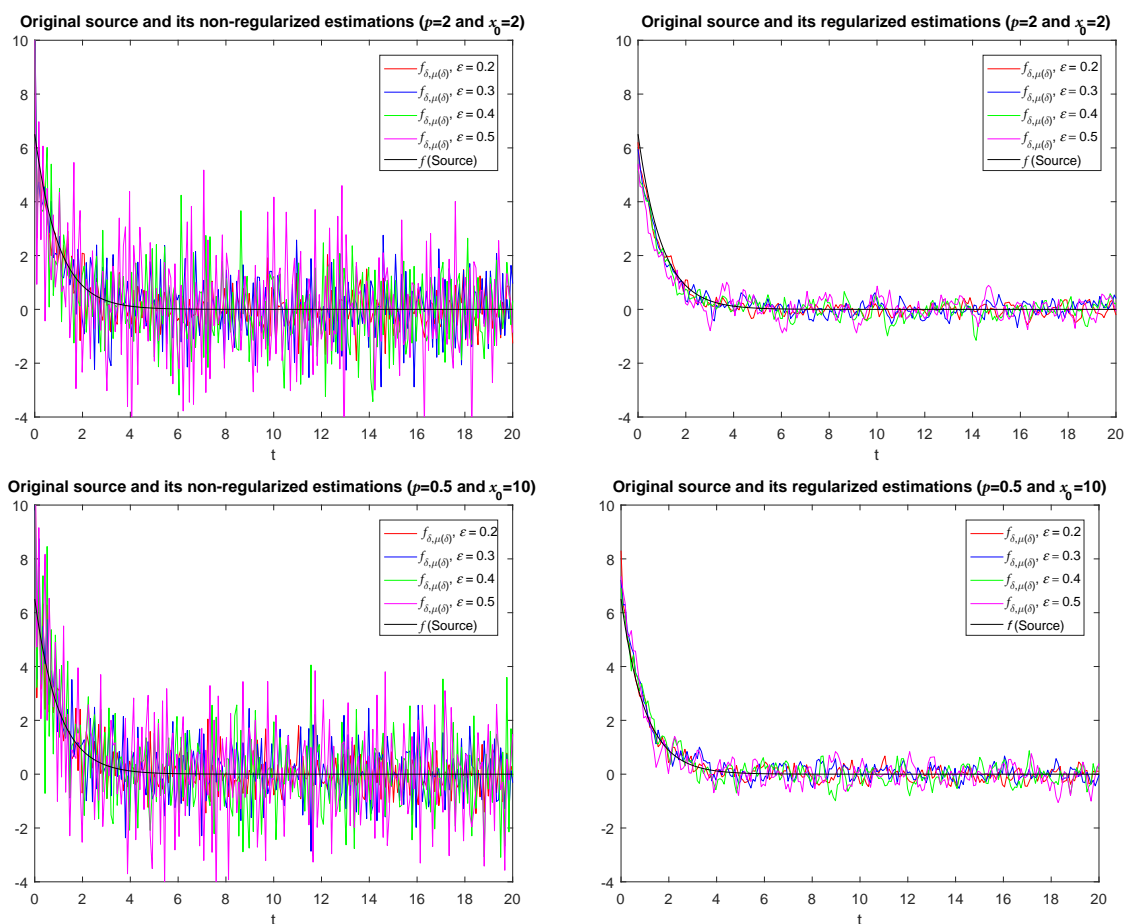


Figure 1: Example 5.1: Original Source and its estimations for the solution  $f_\delta$  (left) and the regularized one  $f_{\delta, \mu}$  (right) for the same set of noise levels.

Figure 1 shows the original source  $f$ , for the example 5.1 along with the non-regularized solution  $f_\delta = Ty_\delta$  (left side) and the regularized one  $f_{\delta, \mu}$  given by (10) (right side). The plots on the top correspond to  $p = 2$  and the data measured at  $x_0 = 2$  and the one below correspond to  $p = 0.5$  and  $x_0 = 10$ . Also, high level noise were considered, we take  $\epsilon \in \{0.2, 0.3, 0.4, 0.5\}$ .

$\epsilon$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$\ f - f_\delta\ $	9.54	9.59	9.62	9.75	10.09	10.17	10.283	10.90	11.85	12.19
$\ f - f_{\delta,\mu}\ $	5.282	5.305	5.307	5.309	5.310	5.315	5.318	5.320	5.326	5.369
Theoretical Bound	6.47	6.75	6.98	7.13	7.46	7.84	8.11	8.30	9.41	10.46

Table 1: Absolute errors for the non-regularized and regularized estimations and theoretical bound for example 5.1 with  $p = 2$  and data measured at  $x_0 = 2$ .

Table 1 contains the absolute errors for the non-regularized and the regularized source solutions when considering  $p = 2$  and measuring position  $x_0 = 2$ .

**Example 5.2** Consider the inverse source problem defined in (1) with modeling parameter values  $\alpha^2 = 0.1, \beta = 0.9, \nu = 1$ . Hence, the equation is given by

$$\begin{cases} u_t(x, t) = 0.1u_{xx}(x, t) - 0.9u_x(x, t) - u(x, t) + f(t), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = 0, & x \in \mathbb{R}, \\ u(0, t) = 0, & t > 0, \\ u(x_0, t) = y(t), & t > 0 \end{cases}$$

with  $f$  given by

$$f(t) = \begin{cases} 0, & 2 > t > 0, \\ 2, & 4 > t \geq 2, \\ -2, & 6 > t \geq 4, \\ 1, & 8 > t \geq 6, \\ -1, & 10 > t \geq 8, \\ 1/2, & 12 > t \geq 10, \\ -1/2, & 14 > t \geq 12, \\ 1/4, & 16 > t \geq 14, \\ -1/4, & 18 > t \geq 16, \\ 0, & t \geq 18. \end{cases} \tag{20}$$

Figure 2 shows the original source  $f$ , for the example 2 along with  $f_\delta$  (left side) and the regularized solution  $f_{\delta,\mu}$  (right side). The plots on the top correspond to  $p = 3$  and the data measured at  $x_0 = 3$  and the ones below correspond to  $p = 0.2$  and  $x_0 = 8$ . For this example, the level noises are  $\epsilon \in \{0.1, 0.15, 0.2, 0.25\}$ .

$\epsilon$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$\ f - f_\delta\ $	13.64	13.72	13.82	13.88	13.97	14.01	14.23	14.61	14.72	15.43
$\ f - f_{\delta,\mu}\ $	2.21	2.85	3.23	3.97	4.31	4.78	4.92	5.03	5.49	5.88
Theoretical Bound	2.37	3.13	3.68	4.13	4.52	4.86	5.17	5.46	5.72	5.97

Table 2: Absolute errors for the non-regularized and regularized estimations and theoretical bound for example 5.2 with  $p = 3$  and data measured at  $x_0 = 3$ .

Table 2 contains the absolute errors for the non-regularized and the regularized source solutions when considering  $p = 3$  and measuring position  $x_0 = 3$ .



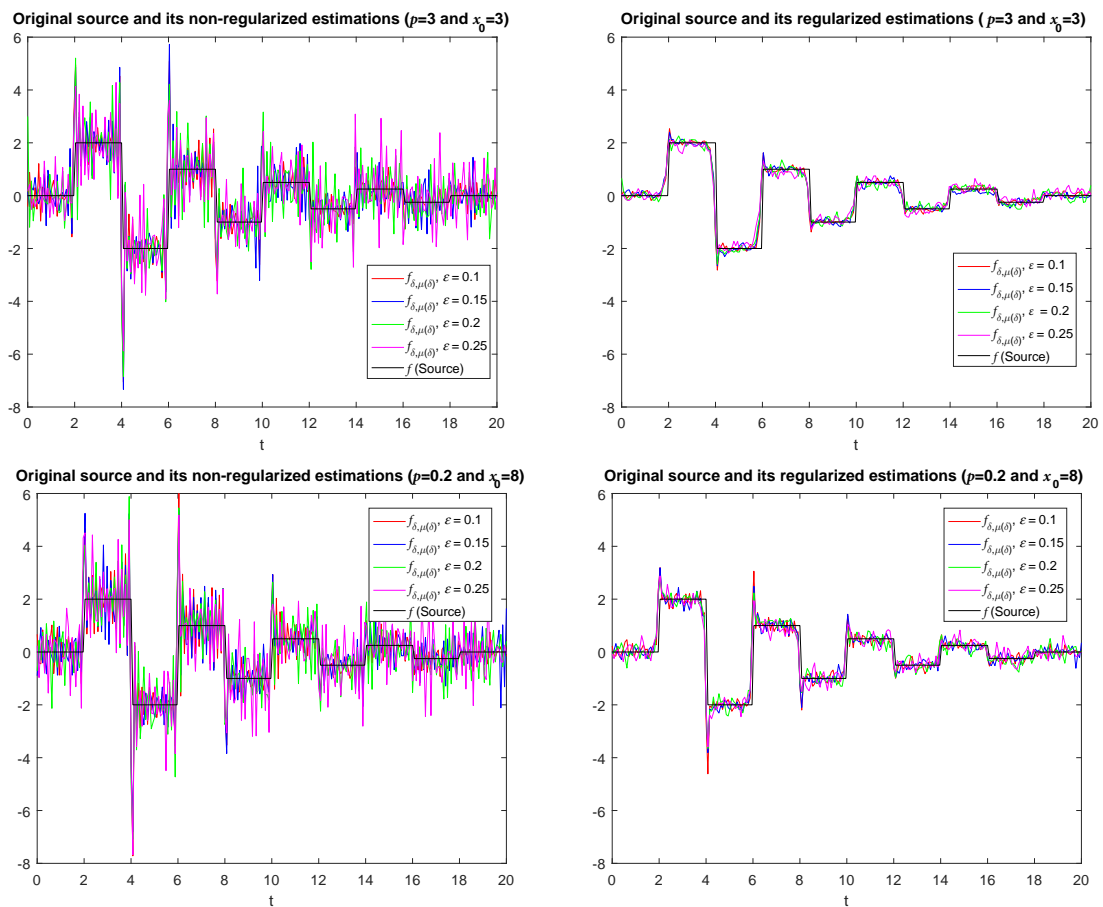


Figure 2: Example 5.2: Original Source and its estimations  $f_{\delta}$  (left) and the regularized solution  $f_{\delta, \mu}$  (right) for the same set of noise levels.

## 6 CONCLUSIONS

We consider the inverse source problem for a 1D transport equation. We define a regularization family of operators to deal with the ill-posedness of the problem by compensation the instability factor in the inverse operator. We proposed a regularization parameter choice rule based on assumption of the noise level in data and the smoothness of the source to be identify. We prove that for the parameter choice rule proposed here, the method is stable and a Hölder type bound for the regularization error is obtained. The numerical examples show an improvement in the regularized solution with the respect to the one obtained when no regularization is applied. In this work we have included few examples where the sources belong to different Hilbert spaces to illustrate the performance of this method. The numerical experiments show good estimates for the source at different noise levels.

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## REFERENCES

- Dou F. and Fu C. Determining an unknown source in the heat equation by a wavelet dual least squares method. *Appl. Math. Lett.*, 22:661–667, 2009.
- Dou F., Fu C., and Yang F.L. Optimal error bound and fourier regularization for identifying an unknown source in the heat equation. *J. Comput. Appl. Math.*, 230(2):728–737, 2009.
- El Badia A. and Ha-Duong T. An inverse source problem in potential analysis. *Inverse Problems*, 16(3):651–663, 2000.
- Engel H., Hanke M., and Neubauer A. *Regularization of Inverse Problems*, volume I. Kluwer Academic Publisher, 1996.
- Fu C. Simplified tikhonov and fourier regularization methods on a general sideways parabolic equation. *J. Comput. Appl. Math.*, 167:449–463, 2004.
- Hadamard J. *Lectures on Cauchy problem in linear Differential Equations*. Yale University Press, New Haven, 1923.
- Hansen C. and O’Leary A.P. The use of the l-curve in the regularization of discrete ill-posed problems. *SIAM J. on Scientific Comp*, 14(6):1487–1503, 1993.
- Johansson B. and Lescnic D. A procedure for determining a spacewise dependent heat source and the initial temperature. *Applicable Analysis*, 87:265–276, 2008.
- Kirsch A. *An introduction to the mathematical theory of inverse problems*. Springer, New York, 2011.
- Li G., Tan Y., Cheng J., and Wang X. Determining magnitude of groundwater pollution sources by data compatibility analysis. *Inverse Prob. Sci. Eng.*, 14:287–300, 2006.
- Macleod R. Widespread intraspecies cross-contamination of human tumor cell lines arising at source. *Int. J. Cancer*, 83(4):555–563, 1999.
- Qian Z., Fu C.L., and Feng X.L. A modified method for high order numerical derivatives. *Appl. Math. Comput.*, 182(2):1191–1200, 2006.
- Qian Z., Fu C.L., and Shi R. A modified method for a backward heat conduction problem. *Appl. Math. Comput.*, 185(1):564–573, 2007.
- Sivergina I., Polis M., and Kolmanovsky I. Source identification for parabolic equations. *Math. Control Signals Syst.*, 16:141–157, 2003.
- Sun N. *Mathematical Model of Groundwater Pollution*. Springer, New York, 1996.
- Trong D., Long N., and Alain P. Nonhomogeneous heat equation: identification and regularization for the inhomogeneous term. *J. of Math. An. and App.*, 312:93–104, 2005.
- Trong D., Quan P., and Alain P. Determination of a two-dimensional heat source: uniqueness, regularization and error estimate. *J. Comput. Appl. Math.*, 191:50–67, 2006.
- Xiao X.L., Heng Z.G., Shi M.W., and Fan Y. Inverse source identification by the modified regularization method on poisson equation. *J. Appl. Math.*, 18:1–13, 2012.
- Yan L., Fu C., and Dou F. A computational method for identifying a spacewise-dependent heat source. *Commun. Numer. Methods Eng.*, 26(5):597–608, 2010.
- Yang F. and Fu C. The method of simplified tikhonov regularization for dealing with the inverse time-dependent heat source problem. *Comput. Math. Appl.*, 60(5):1228–1236, 2010.
- Yang F. and Fu C. Two regularization methods to identify time-dependent heat source through an internal measurement of temperature. *Math. Comput. Modell.*, 53:793–804, 2011.
- Z. Zhao Z., Xie O., Meng Z., and You L. Determination of an unknown source in the heat equation by the method of tikhonov regularization in hilbert scales. *J. App. Math. and Phys.*, 2:10–17, 2014.