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DISCONTINUOUS MIXED SPACE-TIME LEAST-SQUARES FORMULATION FOR TRANSIENT ADVECTION-DIFFUSION-REACTION EQUATIONS

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Abstract.

In a previous work we proposed a constant discontinuous space-time least-squares finite element formulation, where a θ-averaged scheme was used to consider distinct time discretizations and a von Neumann stability analysis displayed, for $\theta \geq 0.5$ unconditionally stable solutions for any Courant number for 1-D problems. Optimal convergence results were obtained for θ = 0.5 and Courant number equal one. In this work we present mixed discontinuous space-time least-square finite element formulations applied for advection-diffusion-reaction equation, resolved into first order system of differential equation approximating both the prime field variable and its fluxes through a θ - averaged scheme to allow distinct time discretizations. We also present coercivity proof of the bilinear form for this problem, together with its error estimates and show that this formulation is not subjected to LBB condition.

1 INTRODUCTION

Numerical approximations are frequently used for solving transport problems modeling phenomena in fluid mechanics and heat transfer among others.

Finite element approximations based on Galerkin semi-discrete formulation has quick loss of accuracy and reduced stability properties. Alternative finite element formulations have been proposed to improve both accuracy and stability, as it occurs with stabilized Galerkin methods as well as with discontinuous space-time Galerkin methods.

An other approach to solve these difficulties is least-squares finite element method which has been applied to stationary problems (Carey 1996) and (Kim 2000) as well as to time-dependent problem with semi-discrete approximations(Carey and Jiang 1987) showing good stability properties and low accuracy (Donea and Quartapelle 1997).

As a contribution to these studies a discontinuous space-time least-squares finite element formulation for hyperbolic equations was developed in (Menna Barreto 1999). The von Neumann stability analysis presented in (Leal-Toledo and Toledo 2001) for 1-D advection problems displayed good stability and accuracy properties in comparison to others formulations mentioned by Donea (1992), although the jump term has destroyed the formulation symmetry.

Even when solving advection-diffusion-reaction equations problems numerical schemes present difficulties around boundaries and internal layers neighborhoods where sharp gradients may appear due to Peclet and/or Damkohler numbers.

In this work we present mixed discontinuous space-time least-squares finite element formulations applied for unsteady advection-diffusion-reaction equation, resolved into first order system of differential equation, approximating both the prime field variable and its fluxes through a θ - averaged scheme to allow distinct time discretizations. We also present coercivity proof of the bilinear form for this problem, together with its error estimates showing that there is no need to satisfying the LBB consistency condition even when using mixed formulations .

2 ADVECTION-DIFFUSION-REACTION EQUATION

2.1 Statement of the problem

Let $\Omega \in \mathbb{R}^m$ be a bounded domain, $m = 2, 3$, with smooth boundary $\Gamma = \Gamma_D \bigcup \Gamma_N$ and Γ_D has positive measure. The unit outward normal vector **n** to Γ is defined almost everywhere. Let $\mathbf{x} = (x_1, ..., x_m) \in \overline{\Omega} = \Omega \cup \Gamma$. The second-order partial differential equation that we want to solve is:

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - \text{div}\,(\mathbf{K}\nabla u) + \sigma u = f \text{ in } \Omega \times (0, T), \tag{1}$$

Equation (1) will be supplied with the boundary condition:

$$\begin{aligned} u &= 0 \text{ on } \Gamma_D \times (0, T), & (2)\\ \mathbf{n} \cdot (-\mathbf{K}\nabla u) &= 0 \text{ on } \Gamma_N \times (0, T), & (3) \end{aligned}$$

and initial condition of the form:

$$u = u^0 \text{ in } \Omega \text{ for } t = 0, \tag{4}$$

where u denotes the unknown quantity being transported by the advective field $\mathbf{v} = (v_1, ..., v_m) \in C^1(\overline{\Omega})^n$, f is a source term, g is the boundary value prescribed for u on Γ_D, h is the boundary value prescribed for u on Γ_N, ∇ is the gradient operator defined as

$\nabla \cdot = \left[\dfrac{\partial \cdot}{\partial x_1} \dots \dfrac{\partial \cdot}{\partial x_m} \right]^T$, reaction coefficient $\sigma(x,t)$ is bounded with $0 \le \sigma(x,t) \le c_1$ and the diffusion coefficient $\mathbf{K} = (a_{ij}(x))_{i,j=1,n}, x \in \overline{\Omega}$, symmetric positive definite and the coefficients a_{ij} are bounded, i.e., there exists positive constantes α_1 and α_2, for all $\xi \in \mathbb{R}^m$ such that:

$$\alpha_1 \xi^T \xi \le \xi^T \mathbf{K} \xi \le \alpha_2 \xi^T \xi. \tag{5}$$

The original equation (1) is resolved into first order differential equations system involving a scalar and a vector field. Introducing the flux $\mathbf{p} = -\mathbf{K}\nabla u$, with $\mathbf{p} = (p_1, ..., p_m)$ and using $\nabla u = -\mathbf{K}^{-1}\mathbf{p}$, we obtain the following first order system for u and $\mathbf{p}$:

$$\frac{\partial u}{\partial t} - \mathbf{v} \cdot \mathbf{K}^{-1}\mathbf{p} + \text{div } \mathbf{p} + \sigma u = f \quad \text{in} \quad \Omega \times (0,T), \tag{6}$$

$$\mathbf{p} + \mathbf{K}\nabla u = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{7}$$

with the boundary condition

$$u = 0 \quad \text{on} \quad \Gamma_D \times (0,T), \tag{8}$$

$$\mathbf{n} \cdot \mathbf{p} = 0 \quad \text{on} \quad \Gamma_N \times (0,T), \tag{9}$$

and initial condition of the form

$$u = u^0 \text{ in } \Omega \text{ for } t = 0. \tag{10}$$

3 DISCONTINUOUS SPACE-TIME MIXED LEAST-SQUARES FORMULATION

We introduce some notations in order to define the space-time least-squares formulation:
Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of interval $I = (0,T)$, and let $I_n = \{(\mathbf{x},t); \mathbf{x} \in \Omega, t_n < t < t_{n+1}\}$ and $\Delta t = t_{n+1} - t_n$ be the n-th interval and local time step, respectively. For each n, we define the space-time integration domain by the strip $Q_n = \Omega \times I_n$ with boundary $\Upsilon_n = \Gamma \times I_n$.

In the n-th space-time strip, the space domain Ω is divided in (Ne) elements $\Omega^1, \dots, \Omega^{Ne}$, satisfying

$$\Omega = \bigcup_{e=1}^{(Ne)} \Omega^e \quad \text{and} \quad \Omega^e \cap \Omega^{e'} = \emptyset \quad \text{if} \quad e \neq e'.$$

The space-time finite elements subspaces are defined by:

$$\begin{aligned}
V_n^h &= \{w^h \in C^0(Q_n), w^h|_{Q_n^e} \in \mathbb{P}_k(Q_n^e);\ w^h = 0 \text{ on } \Upsilon_{Dn}\} \subset H_0^1(Q_n), \\
\widetilde{S}_n^h &= \{\mathbf{p}^h \in [C^0(Q_n)]^m, \mathbf{p}^h|_{Q_n^e} \in [\mathbb{P}_k(Q_n^e)]^m; \mathbf{n} \cdot \mathbf{p}^h = 0 \text{ on } \Upsilon_{Nn}\} \subset [L^2(Q_n)]^m, \\
S_n^h &= \{\mathbf{p}^h \in [C^0(Q_n)]^m, \mathbf{p}^h|_{Q_n^e} \in [\mathbb{P}_k(Q_n^e)]^m; \mathbf{n} \wedge \mathbf{K}^{-1}\mathbf{p}^h = 0 \text{ on } \Upsilon_{Dn} \\
&\quad \text{and } \mathbf{n} \cdot \mathbf{p}^h = 0 \text{ on } \Upsilon_{Nn}\} \subset [L^2(Q_n)]^m.
\end{aligned}$$

where $\Upsilon_{D_n} = \Gamma_D \times I_n$, $\Upsilon_{Nn} = \Gamma_N \times I_n$ and $\mathbb{P}_k$ is the set of piecewise polynomials of degree less than or equal to k continuous in each space-time strip but discontinuous in the interface of different strips.

Consider the following temporal jump operator:

$$[\![u(t)]\!] = u(t^+) - u(t^-) \tag{11}$$

where $u(t_n^{\pm}) = u(\mathbf{x}, t_n^{\pm}) = \lim_{\varepsilon \to 0^{\pm}} u(t_n + \varepsilon)$.

Discontinuous space-time mixed least-squares formulation applied to the problem described by the first order system defined in section (2.1) can be stated as:

Problem P_h : For $n = 1, 2, \cdots, N-1$ find $u^h \in V_n^h$ and $\mathbf{p}^h \in S_n^h$ such that

$$\begin{aligned}&\left(u_t^h + \text{div } \mathbf{p}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{p}^h + \sigma u^h - f, w_t^h + \text{div } \mathbf{q}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\right)_{Q_n} + \\ &+ \left(|[u^h(t_n)]|, \text{div } \mathbf{q}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\right)_{\Omega} + \\ &+ \left(\mathbf{p}^h + \mathbf{K}\nabla u^h, \mathbf{q}^h + \mathbf{K}\nabla w^h\right)_{\Omega} = 0,\end{aligned} \tag{12}$$

for all $(w^h, \mathbf{q}^h) \in V_n^h \times S_n^h$, where the inner products $(\cdot,\cdot)$ are defined as:

$$\begin{aligned}\left(u^h, w^h\right)_{Q_n} &= \int_{t_n}^{t_{n+1}} \int_{\Omega} uw d\Omega dt,\\ \left(u^h, w^h\right)_{\Omega} &= \int_{\Omega} u^h w^h d\Omega,\\ \left(u^h, w^h\right)_{\Upsilon} &= \int_{\Upsilon} u^h w^h\, \mathbf{n}\cdot\mathbf{v} dS.\end{aligned}$$

For simplicity we denote $(.,.)_{\Omega}$ as $(.,.)$.

3.1 Discontinuous space-time least-squares formulations constant in time

Writing the Eq. (12) term coming from the Eq. (6) for any $t = t_{n+\theta}$ $(\theta \in [0,1])$ with

$$\begin{aligned}u^h &= \theta u^h(t_{n+1}^-) + (1-\theta)u^h(t_n^-),\\ \mathbf{p}^h &= \theta \mathbf{p}^h(t_{n+1}^-) + (1-\theta)\mathbf{p}^h(t_n^-).\end{aligned} \tag{13}$$

and assuming finite element approximations for $u^h(t_n)$ and $w^h(t_n)$ constant in time, after time integration we obtain

Problem P_{h_θ}: For $n = 1, 2, \cdots, N-1$ find $u^h \in V_n^h$ and $\mathbf{p}^h \in S_n^h$ such that

$$\widetilde{\mathbb{B}}_\theta\{(u^h, \mathbf{p}^h); (w^h, \mathbf{q}^h)\} = \mathbb{L}_\theta(w^h, \mathbf{q}^h), \quad \forall (w^h, \mathbf{p}^h) \in V_n^h \times S_n^h \tag{14}$$

where

$$\begin{aligned}\widetilde{\mathbb{B}}_\theta\{(u^h, \mathbf{p}^h); (w^h, \mathbf{q}^h)\} \;=\;& \Delta t\theta \left(\text{div } \mathbf{p}^h(t_{n+1}^-) - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{p}^h(t_{n+1}^-) + \sigma u(t_{n+1}^-)\right.,\\ & \left.\text{div } \mathbf{q}^h(t_{n+1}^-) - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h(t_{n+1}^-) + \sigma w^h(t_{n+1}^-)\right) + \\ +\;& \theta\left(u^h(t_{n+1}^-), \text{div } \mathbf{q}^h(t_{n+1}^-) - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h(t_{n+1}^-) + \sigma w^h(t_{n+1}^-)\right)\\ +\;& \left(\mathbf{p}^h(t_{n+1}^-) + \mathbf{K}\nabla u^h(t_{n+1}^-), \mathbf{q}^h(t_{n+1}^-) + \mathbf{K}\nabla w^h(t_{n+1}^-)\right)\end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathbb{L}_\theta(w^h,\mathbf{q}^h) &= \Delta t(\theta-1)\left(\text{div }\mathbf{p}^h(t_n^-)-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{p}^h(t_n^-)+\sigma u(t_n^-)\right., \\ &\quad \left.\text{div }\mathbf{q}^h(t_{n+1}^-)-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h(t_{n+1}^-)+\sigma w^h(t_{n+1}^-)\right)+ \\ &\quad +\theta\left(u^h(t_n^-),\text{div }\mathbf{q}^h(t_{n+1}^-)-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h(t_{n+1}^-)+\sigma w^h(t_{n+1}^-)\right)+ \\ &\quad +\Delta t\left(f(t_{n+\theta}^-),\text{div }\mathbf{q}^h(t_{n+1}^-)-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h(t_{n+1}^-)+\sigma w^h(t_{n+1}^-)\right). \end{aligned}\tag{16}$$

Define the norms:

$$\begin{aligned}||\mathbf{q}^h||^2_{H(\text{div})} &\equiv ||\mathbf{q}^h||^2+||\text{div }\mathbf{q}^h||^2, \\ ||\mathbf{q}^h||^2_{H(\text{curl})} &\equiv ||\mathbf{q}^h||^2+||\text{curl}\mathbf{K}^{-1}\mathbf{q}^h||^2, \\ ||\mathbf{q}^h||^2_{H(\text{div},\text{curl})} &\equiv ||\mathbf{q}^h||^2+||\text{div }\mathbf{q}^h||^2+||\text{curl}\mathbf{K}^{-1}\mathbf{q}^h||^2, \\ ||(w^h,\mathbf{q}^h)||^2_{H^1\times H(\text{div},\text{curl})} &\equiv ||w^h||_1^2+||\mathbf{q}^h||^2_{H(\text{div},\text{curl})}.\end{aligned}$$

Existence and uniqueness for the solution of this problem can be assured by Lax-Milgram lemma with:

Coercivity of $\widetilde{\mathbb{B}}_\theta\{\cdot;\cdot\}$: *There exists a constant* $\widetilde{\gamma}>0$ *such that:*

$$\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\}\geq\widetilde{\gamma}||(w^h(t_{n+1}^-),\mathbf{q}^h(t_{n+1}^-))||^2_{H^1\times H(\text{div})},\ \forall(w^h,\mathbf{q}^h)\in V_n^h\times\widetilde{S}_n^h.\tag{17}$$

Proof. Let β be a constant. By definition of bilinear form (15), adding and subtracting to it the terms $2\beta\Delta t\theta\left(w^h,\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\right)$ and $\Delta t\theta\left((2\beta\sigma-\beta^2)w^h,w^h\right)$, we get:

$$\begin{aligned}\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &= \Delta t\theta\left(\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h+(\sigma-\beta)w^h,\right. \\ &\quad \left.\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h+(\sigma-\beta)w^h\right)+ \\ &+ \left(\mathbf{q}^h+\mathbf{K}\nabla w^h,\mathbf{q}^h+\mathbf{K}\nabla w^h\right)+\Delta t\theta\left((2\beta\sigma-\beta^2)w^h,w^h\right)+ \\ &+ \theta\left(w^h,\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h+\sigma w^h\right)+ \\ &+ 2\beta\Delta t\theta\left(w^h,\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\right)\end{aligned}\tag{18}$$

Integrating by parts and regrouping terms, we obtain:

$$\begin{aligned}\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &= \Delta t\theta\left(\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h+(\sigma-\beta)w^h,\right. \\ &\quad \left.\text{div }\mathbf{q}^h-\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h+(\sigma-\beta)w^h\right)- \\ &- (2\beta\Delta t\theta+\theta)\left(\nabla w^h,\mathbf{q}^h\right)-(2\beta\Delta t\theta+\theta)\left(w^h,\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\right)+ \\ &+ \theta\left(w^h,\sigma w^h\right)+\left(\mathbf{q}^h+\mathbf{K}\nabla w^h,\mathbf{q}^h+\mathbf{K}\nabla w^h\right)+ \\ &+ \Delta t\theta\left((2\beta\sigma-\beta^2)w^h,w^h\right)\end{aligned}\tag{19}$$

Removing the first term in (19), adding and subtracting the terms: $(2\beta\Delta t\theta+1)\left(\nabla w^h,\mathbf{q}^h\right)$, $(2\beta\Delta t\theta+1)\left(\mathbf{K}\nabla w^h,\nabla w^h\right)$ and $(\beta\Delta t\theta+1/2)^2\left(\nabla w^h,\nabla w^h\right)$, we get:

$$\begin{aligned}\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \left(\mathbf{q}^h+(\mathbf{K}-(\beta\Delta t\theta+\theta/2)I)\nabla w^h\right., \\ &\quad \left.\mathbf{q}^h+(\mathbf{K}-(\beta\Delta t\theta+\theta/2)I)\nabla w^h\right)-\end{aligned}$$

$$
\begin{aligned}
&- \quad (2\beta\Delta t\theta + \theta)\left(\nabla w^h, \mathbf{q}^h\right) - (2\beta\Delta t\theta + \theta)\left(w^h, \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\right) + \\
&+ \quad \theta\left(w^h, \sigma w^h\right) + \Delta t\theta\left((2\beta\sigma - \beta^2)w^h, w^h\right) + \\
&+ \quad (2\beta\Delta t\theta + \theta)\left(\nabla w^h, \mathbf{q}^h\right) + (2\beta\Delta t\theta + \theta)\left(\mathbf{K}\nabla w^h, \nabla w^h\right) - \\
&- \quad (\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, \nabla w^h\right) \qquad (20)
\end{aligned}
$$

Adding and subtracting the terms: $(2\beta\Delta t\theta + \theta)\left(\mathbf{q}^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right)$, $(2\beta\Delta t\theta + \theta)\left(\nabla w^h, \mathbf{v}w^h\right)$, $2(\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right)$, $(\beta\Delta t\theta + \theta/2)^2\left((\mathbf{K}^{-1})^T\mathbf{v}w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right)$ and $(2\beta\Delta t\theta + \theta)\left(\mathbf{q}^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right)$ in (20), we have

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \;=\; & \left(\mathbf{q}^h + (\mathbf{K} - (\beta\Delta t\theta + \theta/2)I)\nabla w^h - (\beta\Delta t\theta + \theta/2)w^h(\mathbf{K}^{-1})^T\mathbf{v}\,,\right. \\
& \left.\mathbf{q}^h + (\mathbf{K} - (\beta\Delta t\theta + \theta/2)I)\nabla w^h - (\beta\Delta t\theta + \theta/2)(\mathbf{K}^{-1})^T w^h\mathbf{v}\right) - \\
- \;& (2\beta\Delta t\theta + \theta)\left(\mathbf{q}^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) + \theta\left(w^h, \sigma w^h\right) + \\
+ \;& \Delta t\theta\left((2\beta\sigma - \beta^2)w^h, w^h\right) + (2\beta\Delta t\theta + \theta)\left(\mathbf{K}\nabla w^h, \nabla w^h\right) - \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, \nabla w^h\right) + (2\beta\Delta t\theta + \theta)\left(\mathbf{q}^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) + \\
+ \;& (2\beta\Delta t\theta + \theta)\left(\nabla w^h, w^h\mathbf{v}\right) - 2(\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) - \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left((\mathbf{K}^{-1})^T\mathbf{v}w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) \qquad (21)
\end{aligned}
$$

Removing the first term in (21) and regrouping terms,

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \;\geq\; & 2(\beta\Delta t\theta + \theta/2)\left(\nabla w^h, \mathbf{v}w^h\right) - \\
- \;& 2(\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) + \\
+ \;& 2(\beta\Delta t\theta + \theta/2)\left(\mathbf{K}\nabla w^h, \nabla w^h\right) - (\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, \nabla w^h\right) \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left((\mathbf{K}^{-1})^T\mathbf{v}w^h, (\mathbf{K}^{-1})^T\mathbf{v}w^h\right) + \\
+ \;& 2(\beta\Delta t\theta + \theta/2)\left(w^h, \sigma w^h\right) - \Delta t\theta\beta^2\left(w^h, w^h\right). \qquad (22)
\end{aligned}
$$

Using Green's theorem in first and second terms of (22), we get

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \;\geq\; & (\beta\Delta t\theta + \theta/2)\left(w^h, w^h\left((\beta\Delta\theta + \theta/2)\text{div}\left((\mathbf{K}^{-1})^T\mathbf{v}\right) + 2\sigma\right)\right) - \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left(w^h, w^h\left(\frac{\text{div }\mathbf{v}}{\beta\Delta t\theta + \theta/2} + (\mathbf{K}^{-1})^T\mathbf{v}\cdot(\mathbf{K}^{-1})^T\mathbf{v}\right.\right. \\
+ \;& \left.\left.\frac{\Delta t\beta^2}{(\beta\Delta t\theta + \theta/2)^2}\right)\right) + 2(\beta\Delta t\theta + \theta/2)\left(\mathbf{K}\nabla w^h, \nabla w^h\right) - \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, \nabla w^h\right) + (\beta\Delta t\theta + \theta/2)\left(w^h, w^h\mathbf{v}\cdot\mathbf{n}\right)_{\Upsilon_n} \\
- \;& (\beta\Delta t\theta + \theta/2)^2\left(w^h, w^h(\mathbf{K}^{-1})^T\mathbf{v}\cdot\mathbf{n}\right)_{\Upsilon_n} \qquad (23)
\end{aligned}
$$

Assuming $w^h = 0$ on Υ_{D_n} and $\mathbf{v}\cdot\mathbf{n} < 0$ on Υ_{N_n}, we have

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \;\geq\; & (\beta\Delta t\theta + \theta/2)C_0\left(w^h, w^h\right) - (\beta\Delta t\theta + \theta/2)^2 C_1\left(w^h, w^h\right) + \\
+ \;& 2(\beta\Delta t\theta + \theta/2)\left(\mathbf{K}\nabla w^h, \nabla w^h\right) - (\beta\Delta t\theta + \theta/2)^2\left(\nabla w^h, \nabla w^h\right) \\
+ \;& (\beta\Delta t\theta + \theta/2)^2 C_3\left(w^h, w^h\right)_{\Upsilon_{N_n}} - \\
- \;& (\beta\Delta t\theta + \theta/2)C_2\left(w^h, w^h\right)_{\Upsilon_{N_n}}
\end{aligned}
$$

where

$$
\begin{aligned}
C_0 &= \min\left\{\inf_{x\in\Omega}((\beta\Delta t\theta+\theta/2)\text{div }((\mathbf{K}^{-1})^T\mathbf{v})+2\sigma),0\right\},\\
C_1 &= \sup_{x\in\Omega}\left\{\frac{\text{div }\mathbf{v}}{\beta\Delta t\theta+\theta/2}+(\mathbf{K}^{-1})^T\mathbf{v}\cdot(\mathbf{K}^{-1})^T\mathbf{v}+\frac{\Delta t\beta^2}{(\beta\Delta t\theta+\theta/2)^2}\right\},\\
C_2 &= ||\mathbf{v}\cdot\mathbf{n}||_{L^\infty(\Upsilon_{Nn})}\\
C_3 &= ||(\mathbf{K}^{-1})^T\mathbf{v}\cdot\mathbf{n}||_{L^\infty(\Upsilon_{Nn})}
\end{aligned}
\tag{24}
$$

By Poincaré- Friedrichs and trace inequalities and by (5), follows

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \;\geq\;& (\beta\Delta t\theta+\theta/2)(\alpha_1(C_F^{-1})^2+C_0)\left(w^h,w^h\right)+\\
&(\beta\Delta t\theta+\theta/2)\left\{\alpha_1-C_2C^*-(\beta\Delta t\theta+\theta/2)\left[1+C_1C_F^2\right]\right\}\\
&\left(\nabla w^h,\nabla w^h\right)+(\beta\Delta t\theta+\theta/2)^2C_3\left(w^h,w^h\right)_{\Upsilon_{Nn}}.
\end{aligned}
\tag{25}
$$

Taking $\beta=\dfrac{1}{\Delta t\theta}\left[\dfrac{\alpha_1}{2(1+C_1C_F^2+C_3C^*)}-\dfrac{\theta}{2}\right]$, removing the last terms in (25) and assuming $\delta_1=\alpha_1(C_F^{-1})^2+C_0>0;\quad \delta_2=\frac{\alpha_1}{2}-C_2C^*>0$ we obtain:

$$
\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \;\geq\; \frac{\alpha_1}{2(1+C_1C_F^2+C_3C^*)}\left[\delta_1\left(w^h,w^h\right)+\delta_2\left(\nabla w^h,\nabla w^h\right)\right].\tag{26}
$$

and we have

$$
\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \;\geq\; \delta||w^h||_1^2.\tag{27}
$$

with $\delta=\min\{\delta_1,\delta_2\}$.
To complete this proof we consider $\varepsilon\in\mathbb{R}$ a positive constant. Then,

$$
\varepsilon^2(\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2)<\|w^h\|_1^2 \;\text{ or }\; \varepsilon^2(\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2)\geq\|w^h\|_1^2.
$$

Case 1: $\varepsilon^2(\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2)<\|w^h\|_1^2$:
By (27), we get

$$
\begin{aligned}
\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \;&\geq\; \delta\left(\frac{1}{2}\|w^h\|_1^2+\frac{1}{2}\|w^h\|_1^2\right)\\
&>\; \delta\left(\frac{1}{2}\|w^h\|_1^2+\frac{1}{2}\varepsilon^2\left[\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2\right]\right).
\end{aligned}
\tag{28}
$$

Taking

$$
\gamma_1<\min\left\{\frac{\delta}{2},\frac{\delta}{2}\varepsilon^2\right\},
$$

we have

$$
\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\}>\gamma_1(\|w^h\|_1^2+\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2).
$$

Case 2: $\varepsilon^2(\|\mathbf{q}^h\|^2+\|\text{div }\mathbf{q}^h\|^2)\geq\|w^h\|_1^2$:

Taking a positive real constant η, we have now two possibilities: $\|\mathbf{q}^h\|\leq\eta\|\text{div }\mathbf{q}^h\|$ or $\|\mathbf{q}^h\|>\eta\|\text{div }\mathbf{q}^h\|$.

1. $\|\mathbf{q}^h\| \leq \eta \|\text{div } \mathbf{q}^h\|$:

   Therefore, we have

$$\begin{aligned} \|w^h\|^2 \leq \|w^h\|_1^2 &\leq \varepsilon^2(\|\mathbf{q}^h\|^2 + \|\text{div } \mathbf{q}^h\|^2) \\ &\leq \varepsilon^2(\eta^2\|\text{div } \mathbf{q}^h\|^2 + \|\text{div } \mathbf{q}^h\|^2) \\ &= \varepsilon^2(\eta^2+1)\|\text{div } \mathbf{q}^h\|^2. \end{aligned} \tag{29}$$

   Taking $\eta < 1$, we obtain,

$$\|w^h\|^2 \;<\; 4\varepsilon^2\|\text{div } \mathbf{q}^h\|^2. \tag{30}$$

   and by (15), we obtain

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \Delta t\theta\|\text{div } \mathbf{q}^h\|^2 - 2\Delta t\theta\|\text{div } \mathbf{q}^h\|(\| - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\|) + \\ &+ \Delta t\theta\| - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\|^2 + \theta(\sigma w^h, w^h) - \\ &- \theta\|w^h\|\|\text{div } \mathbf{q}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\| + \|\mathbf{q}^h + k\nabla w^h\|^2 \end{aligned} \tag{31}$$

   and

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \Delta t\theta\|\text{div } \mathbf{q}^h\|^2 - 2\Delta t\theta\|\text{div } \mathbf{q}^h\|(\|\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\| + \|\sigma w^h\|) + \\ &+ \Delta t\theta\| - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\|^2 - \\ &- \theta\|w^h\|\|\text{div } \mathbf{q}^h\| - \theta\|w^h\|\|\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\|. \end{aligned} \tag{32}$$

   Taking $C_3 \geq 0$ such that $\|\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\| \leq C_3\|\mathbf{q}^h\|$,

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \Delta t\theta\|\text{div } \mathbf{q}^h\|^2 - \theta\|w^h\|C_3\|\mathbf{q}^h\| - \\ &- \theta\|\text{div } \mathbf{q}^h\|\left(2\Delta tC_3\|\mathbf{q}^h\| + 2\Delta t\|\sigma\|\|w^h\| + \|w^h\|\right) \end{aligned} \tag{33}$$

   and

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \|\text{div } \mathbf{q}^h\|^2\left(\Delta t\theta - 2\theta\varepsilon\eta C_3 - 2\theta(\eta\Delta tC_3 + 4\varepsilon\Delta t\|\sigma\| + 2\varepsilon)\right) \\ &\geq \gamma_2\|\text{div } \mathbf{q}^h\|^2, \end{aligned} \tag{34}$$

   where

$$\gamma_2 = \left(\Delta t\theta - 2\theta\varepsilon\eta C_3 - 2\theta(\eta\Delta tC_3 + 4\varepsilon\Delta t\|\sigma\| + 2\varepsilon)\right).$$

   Thus, using (27),

$$\begin{aligned} 2\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \delta||w^h||_1^2 + \gamma_2\|\text{div } \mathbf{q}^h\|^2 \\ &> \delta||w^h||_1^2 + \frac{\gamma_2}{2}\|\text{div } \mathbf{q}^h\|^2 + \frac{\gamma_2}{2\eta}\|\mathbf{q}^h\|^2. \end{aligned}$$

   Taking

$$\gamma_3 \leq \min\left\{\frac{\delta}{2}, \frac{\gamma_2}{4}, \frac{\gamma_2}{4\eta}\right\},$$

   we get

$$\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \geq \gamma_3(\|\mathbf{q}^h\|^2 + \|\text{div } \mathbf{q}^h\|^2 + \|w^h\|_1^2).$$

2. $\|\mathbf{q}^h\| > \eta\|\text{div }\mathbf{q}^h\|$:

   Inthis case we have

$$\begin{aligned} \|w^h\|^2 \leq \|w^h\|_1^2 &\leq \varepsilon^2(\|\mathbf{q}^h\|^2 + \|\text{div }\mathbf{q}^h\|^2) \\ &\leq \varepsilon^2\left(\|\mathbf{q}^h\|^2 + \frac{1}{\eta^2}\|\mathbf{q}^h\|^2\right) \\ &\leq \frac{\varepsilon^2(\eta^2+1)}{\eta^2}\|\mathbf{q}^h\|^2. \end{aligned} \tag{35}$$

   Taking $\eta < 1$, we obtain,

$$\|w^h\|^2 < 4\frac{\varepsilon^2}{\eta^2}\|\mathbf{q}^h\|^2 \tag{36}$$

   Hence, $\|w^h\| < 2(\varepsilon/\eta)\|\mathbf{q}^h\|$. Similarly, $\|\nabla w^h\| < 2(\varepsilon/\eta)\|\mathbf{q}^h\|$. By definition (15), we obtain

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \Delta t\theta\|\text{div }\mathbf{q}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h + \sigma w^h\|^2 + \\ &+ \theta(\sigma w^h, w^h) - \theta\left(w^h, \text{div }\mathbf{q}^h - \mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\right) + \\ &+ \|\mathbf{q}^h\|^2 - 2(\mathbf{q}^h, \mathbf{K}\nabla w^h) + \|\mathbf{K}\nabla w^h\|^2 \\ &\geq \|\mathbf{q}^h\|^2 - \theta\|w^h\|\left(\|\text{div }\mathbf{q}^h\| + \|\mathbf{v}\cdot\mathbf{K}^{-1}\mathbf{q}^h\|\right) - \\ &- 2\|\mathbf{q}^h\|\|\mathbf{K}\nabla w^h\|. \end{aligned} \tag{37}$$

   Taking $C_4 \geq 0$ such that $\|\mathbf{K}\nabla w^h\| \leq C_4\|\nabla w^h\|$, and C_3 as above,

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \|\mathbf{q}^h\|^2 - \theta\|w^h\|\left(\|\text{div }\mathbf{q}^h\| + C_3\|\mathbf{q}^h\|\right) - \\ &- 4C_4\|\mathbf{q}^h\|\|\nabla w^h\| \end{aligned} \tag{38}$$

   or

$$\begin{aligned} \widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \|\mathbf{q}^h\|^2\left(1 - 2\theta\left(\frac{\varepsilon}{\eta^2} + \frac{C_3}{\eta}\right) - 2C_4\frac{\varepsilon}{\eta}\right) \\ &\geq \gamma_4\|\mathbf{q}^h\|^2, \end{aligned} \tag{39}$$

   where

$$\gamma_4 = \left(1 - 2\theta\left(\frac{\varepsilon}{\eta^2} + \frac{\varepsilon C_3}{\eta}\right) - 4C_4\frac{\varepsilon}{\eta}\right).$$

   Using (27) again,

$$\begin{aligned} 2\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} &\geq \delta||w^h||_1^2 + \gamma_4\|\mathbf{q}^h\|^2 \\ &> \delta||w^h||_1^2 + \frac{\eta^2\gamma_4}{2}\|\text{div }\mathbf{q}^h\|^2 + \frac{\gamma_4}{2}\|\mathbf{q}^h\|^2. \end{aligned} \tag{40}$$

   Taking

$$\gamma_5 \leq \min\left\{\frac{\delta}{2}, \frac{\eta^2\gamma_4}{4}, \frac{\gamma_4}{4}\right\},$$

   we get

$$\widetilde{\mathbb{B}}_\theta\{(w^h,\mathbf{q}^h);(w^h,\mathbf{q}^h)\} \geq \gamma_5(\|\mathbf{q}^h\|^2 + \|\text{div }\mathbf{q}^h\|^2 + \|w^h\|_1^2). \tag{41}$$

Taking $\widetilde{\gamma} = \min\{\gamma_1, \gamma_3, \gamma_5\}$, we obtain

$$\widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \geq \widetilde{\gamma}||(w^h(t_{n+1}^-), \mathbf{q}^h(t_{n+1}^-))||^2_{H^1 \times H(\text{div })}, \tag{42}$$

for all $(w^h, \mathbf{q}^h) \in V_n^h \times \widetilde{S}_n^h$.

□

To obtain a better approximation to the flow, we can add to system (6-7) the equation

$$\text{curl}(\mathbf{K}^{-1}\mathbf{p}) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{43}$$

with boundary condition

$$\mathbf{n} \wedge \mathbf{K}^{-1}\mathbf{p} = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \tag{44}$$

where $\wedge$ denotes the exterior vector product and obtain:

Problem $P1_{h_\theta}$: For $n = 1, 2, \cdots, N-1$ find $u^h \in V_n^h$ and $\mathbf{p}^h \in S_n^h$ such that

$$\mathbb{B}_\theta\{(u^h, \mathbf{p}^h); (w^h, \mathbf{q}^h)\} = \widetilde{\mathbb{B}}_\theta\{(u^h, \mathbf{p}^h); (w^h, \mathbf{q}^h)\} + \left(\text{curl}(\mathbf{K}^{-1}\mathbf{p}^h), \text{curl}(\mathbf{K}^{-1}\mathbf{q}^h)\right) \tag{45}$$

Existence and uniqueness for the solution of this problem can be assured by:

Continuity of $\mathbb{B}_\theta\{\cdot;\cdot\}$: *There exists a constant* $M < \infty$, *such that*

$$|\mathbb{B}_\theta\{(u^h, \mathbf{p}^h); (w^h, \mathbf{q}^h)\}| \leq M||(u^h, \mathbf{p}^h)||_{H^1 \times H(\text{div ,curl})}||(w^h, \mathbf{q}^h)||_{H^1 \times H(\text{div ,curl})}, \tag{46}$$

for all $(u^h, \mathbf{p}^h), (w^h, \mathbf{q}^h) \in V_n^h \times S_n^h$.

Coercivity of $\mathbb{B}_\theta\{\cdot;\cdot\}$: *There exists a constant* $\gamma > 0$ *such that:*

$$\mathbb{B}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \geq \gamma||(w^h(t_{n+1}^-), \mathbf{q}^h(t_{n+1}^-))||^2_{H^1 \times H(\text{div ,curl})}, \tag{47}$$

for all $(w^h, \mathbf{q}^h) \in V_n^h \times S_n^h$.

Proof: By definition of bilinear form (45), we have

$$\mathbb{B}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} = \widetilde{\mathbb{B}}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} + (\text{curl}(\mathbf{K}^{-1}\mathbf{q}^h), \text{curl}(\mathbf{K}^{-1}\mathbf{q}^h)) \tag{48}$$

Thus, by (17), we have

$$\mathbb{B}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \geq \widetilde{\gamma}||(w^h(t_{n+1}^-), \mathbf{q}^h(t_{n+1}^-))||^2_{H^1 \times H(\text{div })} + ||\text{curl}(\mathbf{K}^{-1}\mathbf{q}^h)||^2_{H(\text{curl})} \tag{49}$$

Then there exists $\gamma > 0$, such that

$$\mathbb{B}_\theta\{(w^h, \mathbf{q}^h); (w^h, \mathbf{q}^h)\} \geq \gamma||(w^h(t_{n+1}^-), \mathbf{q}^h(t_{n+1}^-))||^2_{H^1 \times H(\text{div ,curl})}. \tag{50}$$

□

Consistency: The exact solution satisfies the discrete formulation

$$\mathbb{B}_\theta\{(u,\mathbf{p});(w^h,\mathbf{q}^h)\} = \mathbb{L}_\theta(w^h,\mathbf{q}^h), \quad \forall (w^h,\mathbf{q}^h) \in V_n^h \times S_n^h. \tag{51}$$

Orthogonality of the error:

$$\mathbb{B}_\theta\{(u-u^h,\mathbf{p}-\mathbf{p}^h);(w^h,\mathbf{q}^h)\} = 0, \quad \forall (w^h,\mathbf{q}^h) \in V_n^h \times S_n^h. \tag{52}$$

Proof. This is a direct consequence of the bilinearity of $\mathbb{B}_\theta(.,.)$ and of the consistency previously shown.

Assumption A: The domain Ω is convex or the boundary Γ is of class $C^{1,1}$.
Error estimates for this problem can be found, considering $V_h^l(\Omega) \subset H_1$ and $S_h^k(\Omega) \subset L^2$ spaces of C^0 piecewise polynomial finite element interpolations of degree l and k respectively. From general finite element approximation theory and from Assumption A we have the estimates:

$$||u-\widetilde{u}^h|| + h||u-\widetilde{u}^h||_1 \leq Ch^{l+1}||u||_{l+1} \tag{53}$$

$$||\mathbf{p}-\widetilde{\mathbf{p}}^h|| + h||\mathbf{p}-\widetilde{\mathbf{p}}^h||_{H(\text{div},\text{curl})} \leq Ch^{k+1}||\mathbf{p}||_{k+1} \tag{54}$$

where $\widetilde{u}^h$ and $\widetilde{\mathbf{p}}^h$ are the standard finite element interpolant of u and $\mathbf{p}$.

Theorem: Let Ω satisfy Assumption A. If $(u^h,\boldsymbol{p}^h)$ satisfies (12) and $(u,\boldsymbol{p})$ satisfies (6), then

$$||(u-u^h,\mathbf{p}-\mathbf{p}^h)||_{H^1\times H(\text{div},\text{curl})} \leq C\left(h^l||u||_{l+1} + h^k||\mathbf{p}||_{k+1}\right). \tag{55}$$

Proof. From the coercivity (47) and the orthogonality property (52), we have

$$\begin{aligned}
\gamma||(u-u^h,\mathbf{p}-\mathbf{p}^h)||^2_{H^1\times H(\text{div},\text{curl})} &\leq \mathbb{B}_\theta\{(u-u^h,\mathbf{p}-\mathbf{p}^h);(u-u^h,\mathbf{p}-\mathbf{p}^h)\} \\
&\leq \mathbb{B}_\theta\{(u-u^h,\mathbf{p}-\mathbf{p}^h);(u-\widetilde{u}^h,\mathbf{p}-\widetilde{\mathbf{p}}^h)\} + \\
&+ \mathbb{B}_\theta\{(u-u^h,\mathbf{p}-\mathbf{p}^h);(\widetilde{u}^h-u^h,\widetilde{\mathbf{p}}^h-\mathbf{p}^h)\} \\
&\leq \mathbb{B}_\theta\{(u-u^h,\mathbf{p}-\mathbf{p}^h);(u-\widetilde{u}^h,\mathbf{p}-\widetilde{\mathbf{p}}^h)\}.
\end{aligned} \tag{56}$$

By (46), we obtain

$$\begin{aligned}
\gamma||(u-u^h,\mathbf{p}-\mathbf{p}^h)||^2_{H^1\times H(\text{div},\text{curl})} &\leq M||(u-u^h,\mathbf{p}-\mathbf{p}^h)||_{H^1\times H(\text{div},\text{curl})} \times \\
&\quad ||(u-\widetilde{u}^h,\mathbf{p}-\widetilde{\mathbf{p}}^h)||_{H^1\times H(\text{div},\text{curl})}.
\end{aligned} \tag{57}$$

Thus

$$||(u-u^h,\mathbf{p}-\mathbf{p}^h)||_{H^1\times H(\text{div},\text{curl})} \leq \frac{M}{\gamma}||(u-\widetilde{u}^h,\mathbf{p}-\widetilde{\mathbf{p}}^h)||_{H^1\times H(\text{div},\text{curl})}. \tag{58}$$

Therefore, by (53) and (54)

$$||(u-u^h,\mathbf{p}-\mathbf{p}^h)||_{H^1\times H(\text{div},\text{curl})} \leq C\left(h^l||u||_{l+1} + h^k||\mathbf{p}||_{k+1}\right) \tag{59}$$

□

We can use $k = l$, which is a convenient choice from the computational point of view and this estimates holds independently of any mesh parameter.

4 CONCLUSION

For the diffusion-advection-reaction problem we derived and presented here a mixed discontinuous space-time least-squares formulation proving its solution existence and uniqueness even when using equal order interpolations for both fields. Besides that adding to this formulation the curl equation we were able to obtain H_1 norm error estimates for scalar and vector approximated fields.

REFERENCES

A.L. Brooks and T. J. R. Hughes. Streamline Upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the Incompressible Navier-Stokes equations, *Computer Methods in Applied Mechanics and Engineering*, 32:199-259, 1982.

G.F. Carey and B.N. Jiang. Least-Squares finite elements for first order hyperbolic systems, *International Journal for Numerical Methods in Engineering*, 24:1-13, 1987.

R.G.S. Castro. Análise Numérica de Formulações de Elementos Finitos Espaço Tempo para Escoamentos Miscíveis, *Ph.D. thesis, Programa de Engenharia Civil, COPPE/UFRJ, Rio de Janeiro, Brasil*, 1999.

R. Codina. Comparison of some finite element methods for solving the diffusion-convection-reaction equation, *Computer Methods in Applied Mechanics and Engineering*, 156:185-210, 1998.

J. Donea and L. Quartapelle. An introduction to finite element methods for transient advection problems, *Computer Methods in Applied Mechanics and Engineering*, 96:169-203, 1992.

T.J. Hughes and H.M. Gregory. Space-time finite element methods for elastodynamics: formulations and error estimates, *Computer Methods in Applied Mechanics and Engineering*, 66:339-363, 1988.

B. Jiang. *The Least-Squares Finite Element Method*, Springer Verlag, 1998.

C. Johnson. Numerical Solution of Partial Differential Equations by the Finite Element Method, *Cambridge Univ. Press, Cambridge*, 1987.

C. Johnson, U. Navert and J. Pitkäranta. Finite element methods for linear hyperbolic problems, *Computer Methods in Applied Mechanics and Engineering*, 45:285-312, 1984.

Y. Kim and S. Lee. Least-squares mixed method for second-order elliptic problems, *Applied Mathematics and Computation*, 115:89-100, 2000.

R.C.P. Leal-Toledo and E. M. Toledo. Least-squares finite element for 2-D advective transport modeling, *Technical Report - LNCC*, 53, 2001.

R.C.P. Leal-Toledo and E.M. Toledo. A finite element least-squares formulation for stationary and transient advection-diffusion problem, *Proceedings of 52^o Seminário Brasileiro de Análise, São José dos Campos, São Paulo, SP, Brasil*, 2000.

R. C. P. Leal-Toledo, M. F. S. Menna Barreto and E. M. Toledo. A finite element least-squares formulation for stationary and transient advection-diffusion problem, *Proceedings of XX Iberian Latin-American Congress on Computational Methods in Engineering, São Paulo, SP, Brasil*, 1999.

R.C.P. Leal-Toledo and E. M. Toledo. Least-squares finite element for 2-D Advective transport modeling, *Proceedings of XXII Iberian Latin-American Congress on Computational Methods in Engineering, Campinas, SP, Brasil*, 2001a.

G. Lube and G. Rapin. Residual-based stabilized higher-order FEM for advection-dominated problems,*Computer Methods in Applied Mechanics and Engineering*, 195:4124-4138, 2006.

M.F.S.S. Menna Barreto and R. C. P. Leal-Toledo. Formulação de mínimos quadrados aplicada ao problema de advecção-difusão, *Proceedings of 48^o Seminário Brasileiro de Análise, LNCC, Petrópolis, Rio de Janeiro, Brasil*, 1998.

M.F.S.S. Menna Barreto. Formulação de Mínimos Quadrados Aplicada Problemas de Advecção-Difusão, *M.Sc. thesis, Departament of Computer Science, PGCC/UFF, Niterói, Brasil*, 1999.

U. Navert. A Finite Element Method for Convection-Diffusion Problems, *Ph.D. thesis, Departament of Computer Science, Chalmers University of Technology, Goteborg, Sweden*, 1982.

H. Nguyen and J. Reynen. A space-time least-squares finite element scheme for advection-diffusion equations, *Comput. Methods Appl. Mech. Engngr.*, 42:331- 342, 1984.

A.I. Pehlivanov, G.F. Carey and P.S. Vassilevski. Least-square mixed finite element methods for non-selfadjoint elliptic problems: I. Error estimatives, *Numerische Mathematik*, 72:501-522, 1996.