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SEMI-ANALYTICAL SOLUTION OF NON-LINEAR VIBRATION OF BEAMS WITH CLEARANCES CONSIDERATION

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Abstract. The semi-analytical solution of the nonlinear vibrational behavior of beams with clearance is herein presented. A slender beam, clamped in its upper extreme models a drillstring constrained to move inside an outer cylinder (the borehole wall) with clearance which adds a strong nonlinearity to the problem. In a first simplification a Bernoulli-Euler beam of a Hookean material and uniform cross section is subjected to self weight and an axial load at the bottom end to simulate the static reaction force when the drill bit touches the formation. After the initial configuration is attained, the beam is considered clamped at the top and hinged at the bottom. Also a dynamic perturbation moment is applied at the beam bottom. The energy contribution of the clearance nonlinearity due to the discontinuous contact between the beam and the borehole wall is taken into account by means of a spring and represented by power series expansion in the lateral displacements. The solution is found by first applying a direct method with extended trigonometric series that ensure the uniform convergence of the basic unknown functions in the spacial domain. The resulting nonlinear differential system in the time variable is then solved by a standard integration scheme. Deformed configurations at different times are reported. Additionally the static state analysis and the critical load of the prestressed beam are included.

1 INTRODUCTION

As is known a flexible beam subjected to axial loads may undergo of a geometrical stiffening or softening effect, depending on the existence of tensile or compressive forces ((Sharf, 1995)). Some non-linear model of vibroimpacting structures with non-linear strain-displacement relationships are summarized and compared by (Trindade and Sampaio, 2002) and a finite element approach is proposed. Relating drillstrings, several works have been published. (Yigit and Christoforous, 1996) developed a model to study the transverse vibrations of drillstrings caused by axial loading and impact with the bore hole. The Assumed Modes Methods is employed to find the governing equations of motion. In particular a one-mode approximation is used. Other works dealing with more complex models (non-linearities, diverse couplings) have been published (e.g. a lumped mass model is dealt with in (Yigit and Christoforous, 1998) and an integrated mathematical model((Tucker and Wang, 1999)). In the present paper the semi-analytical solution of a vertical rod subjected to variable normal force (due to self-weight and an axial compressive reaction, prevailing tensile stress) restricted to move within a spatial domain with clearances (backlash), is found by means of a variational approach. The dynamic behavior of a uniform, very slender beam subjected to a dynamic external bending moment at its lower end is strongly modified when the response —$v(x,t)$— is larger than the clearance and receives an elastic reaction from the boundary. Although in this first study the equation of motion to be solved by a direct method using a complete set (Whole Element Method, WEM, (Rosales, 1997; Rosales and Filipich, 2002)) is linear-bending, a nonlinear problem has to be solved due to the "Winkler reaction" that only acts when the $v(x,t)$ magnitude exceeds the limit of the fixed clearance. The axial/bending coupling is not considered at this stage though the stiffness loss known as "geometric stiffening" effect is taken into account. The nonlinear response of the "Winkler" soil is simulated with a integer power series expansion in the *elastica* $v(x,t)$. This study is a complement of the ideas presented by (Trindade et al., 2005) in which the nonlinear oscillations of drillstrings used for oilwell drilling is addressed through the finite element method. In this paper a nonlinear problem with axial/bending coupling is tackled and boundary elastic reaction is applied when the nodal unknowns surpass the clearance. In the present work, the reaction of the nonlinear stiffness spring is included in the equation of motion. As may be observed, the analogy is not complete and the authors are, at present, comparing the response with nonlinear contributions. Unfortunately the results are not available at the time this paper is written but the authors expect to show them at the ENIEF 2006.

2 MODEL FORMULATION

Let us analyze the bar depicted in Figure 1 subjected to self-weight that induces a tensile state and a reaction R at its lower end that inputs a uniform compression. This load system gives place to an axial stress

$$\begin{aligned}\sigma_0 &= \sigma_0(X) \text{ that is}\\ \sigma_0(X) &= \gamma(L-X)-\frac{R}{A}\end{aligned} \tag{1}$$

where γ is the unit weight and A is the cross-sectional area. If parameters $r \equiv R/\gamma AL$ and $c = 1-r$ are introduced (see Fig. 1) the stress σ_0 may be re-written as

$$\sigma_0(X) = \gamma L(1-r-\frac{X}{L}) \tag{2}$$

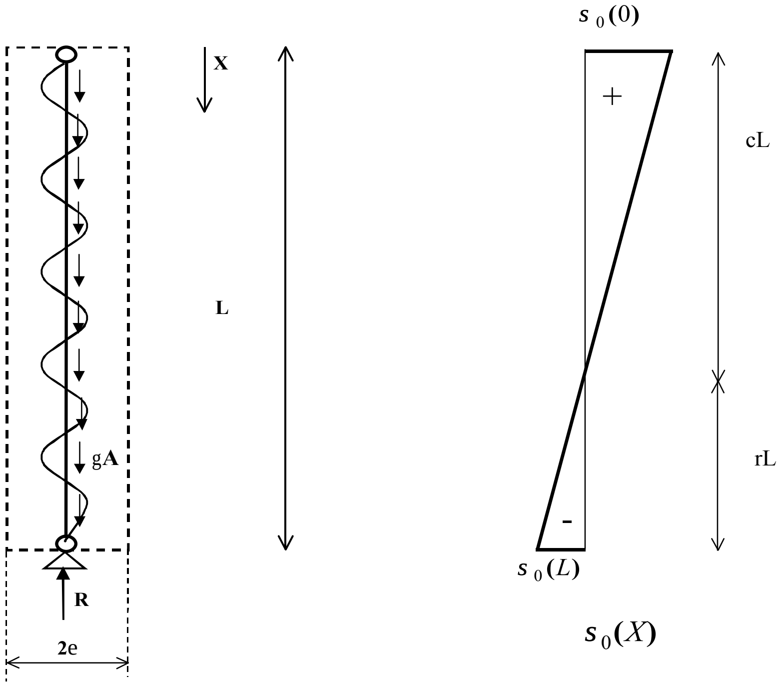

Figure 1: Vertical rod subjected to self weight and end reaction restrained to move in an outer cylinder and initial stress diagram.

From this prestressed state the equation of motion of the rod of Figure 1 subjected to a dynamic moment $M(t)$ applied at $X = L$. Then let us define the *Lagrangian* function $\mathcal{L}$ as

$$\mathcal{L} = U + U_R + U_e - T \tag{3}$$

in which U is the elastic strain energy, U_R is the ”Winkler soil” energy, U_e is the energy due to the load position and T is the kinetic energy, that arise from a generic deformation $v = \hat{v}(X,t)$. Then if E is the Young's modulus and J is the moment of inertia of the uniform section of the rod the following expressions yield

$$\begin{aligned} 2U &= EJ\int_0^L \left(\frac{d^2\hat{v}}{dX^2}\right)^2 dX + \int_V \sigma_0(X)\left(\frac{d\hat{v}}{dX}\right)^2 dXdA \\ U_R &= \int_0^L U_R^*(X,t)dX \qquad U_e = -M_0(t)\frac{d\hat{v}(L)}{dX} \\ 2T &= \rho A\int_0^L \left(\frac{d\hat{v}(L)}{dX}\right)^2 dX \end{aligned} \tag{4}$$

$U_R^*(X,t)$ is the energy per unit of length due to the nonlinear deformation of the medium where the beam dynamically bends. That is, if 2ϵ denotes the width of the empty region, the reaction $F(v)$ is as shown in Figure 2

The slope $\tan\alpha$ depends on the stiffness k_0 of the surrounding soil and then

$$F(v) = \begin{cases} k_0 v(X,t), |v(X,t)| < \epsilon \\ 0, |v(X,t)| \leq \epsilon \end{cases} \tag{5}$$

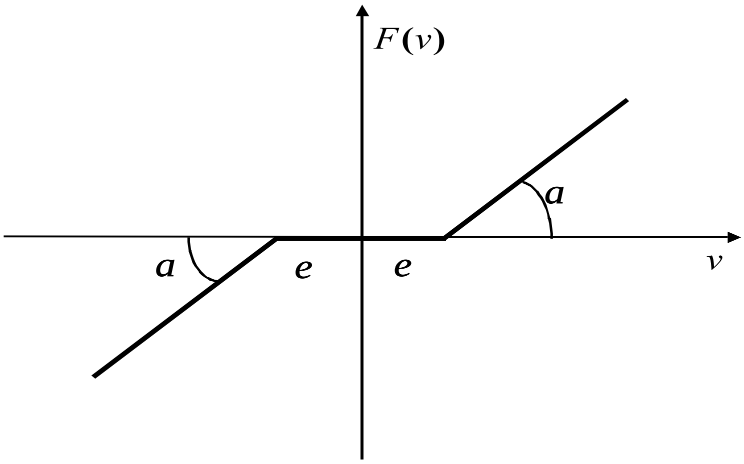

Figure 2: Response force model of the discontinuous soil reaction.

In this work the following series for $F(v)$ (origin of the strong nonlinearity)

$$F(v) = k_0 \sum_{n=1,3,5,\cdots}^{MG} G_n v^n(X,t) \tag{6}$$

The G_n's are found by a least squares with which the unit energy yields

$$U_R^* = k_0 \sum_{n=1,3,5,\cdots} G_n \frac{v^{n+1}(X,t)}{n+1} \tag{7}$$

Twelve terms were enough to yield the curve shown in Figure 3 Now it is possible to apply Hamilton's Principle as follows

$$\delta \mathcal{F}[v] = 0 \tag{8}$$

where

$$\mathcal{F} = \int_{t_1}^{t_2} \mathcal{L} dt$$

The equation of motion is derived after the introduction of a proposed solution that linearly combines a complete subset which satisfies the essential boundary conditions (BC).

2.1 WEM Solution

The following extended trigonometric series is proposed for the dynamic response $v(X,t)$

$$v(X,t) = \sum_{i=1}^{M} A_{1i}(t) \sin(\frac{i\pi X}{L}) + \frac{X}{L} A_{10}(t) + \alpha_{10}(t) \tag{9}$$

Such series is, as is known, e.g. (Rosales, 1997; Filipich and Rosales, 2000; Rosales and Filipich, 2002), uniformly convergent to the classical solution in the domain $0 \leq X \leq L$; analogously the same property holds for the first derivative. Meanwhile the second derivative converges in L_2

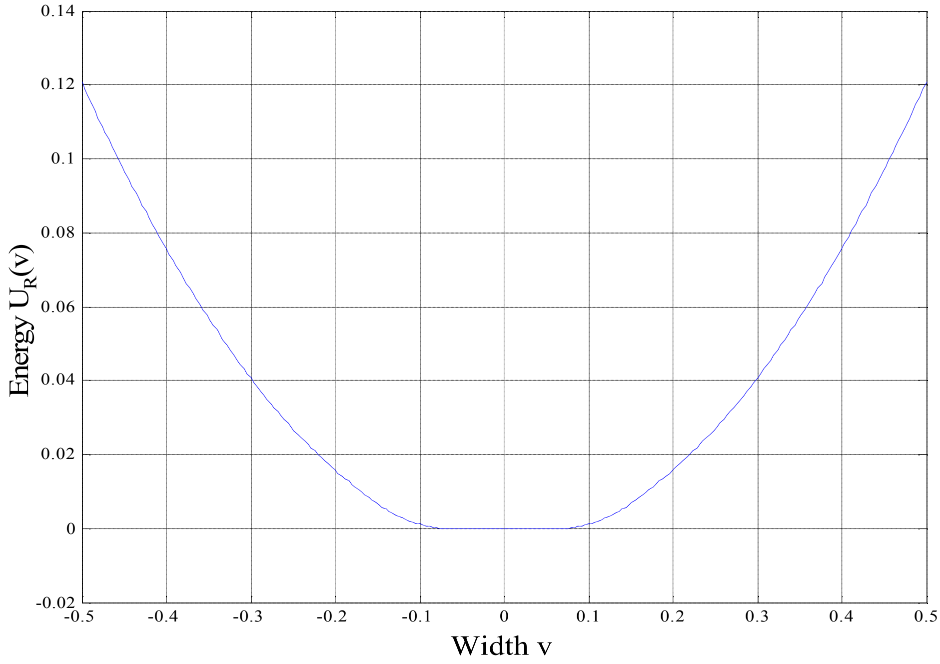

Figure 3: Energy resulting from the discontinuous soil reaction. MG=12

$$Lv'(X,t) = \sum_{i=1}^{M} A_{1i}(t)\cos(\frac{i\pi X}{L}) + A_{10}(t) \tag{10}$$

$$L^2 v''(X,t) = -\sum_{i=1}^{M} \gamma_i^2 A_{1i}(t)\sin(\frac{i\pi X}{L}) \tag{11}$$

where $\gamma_n \equiv n\pi$. Now, the statement of Hamilton's principle requires of convergence in the mean (in L_2) of the proposed series solution and the satisfaction of the geometric or stable BC. If the WEM series do not satisfy identically the essential BC, they will have to be considered for instance, by means of the well-known Lagrange multipliers. Given the large slenderness of the studied rod, the qualitative difference between a clamped and a hinged end may be considered negligible. Consequently it will assumed that the beam is hinged at both ends ($X = 0$ and $X = L$) and the BC are

$$v(0,t) = 0; \qquad v(L,t) = 0 \tag{12}$$

from which

$$A_{10}(t) = 0; \qquad \alpha_{10}(t) = 0 \tag{13}$$

Then in this case, evidently the natural BC are also identically verified. Thus the series result

$$v(X,t) = \sum_{i=1}^{M} A_{1i}(t)\sin(\frac{i\pi X}{L}) \tag{14}$$

2.2 Nonlinear spring representation

Before the detailed derivation of the equation of motion, the expression of $F(v)$ (or U_R^*) is introduced in this Subsection. Let $v^m(X,t)$ be the m^{th} power of the dynamic *elastica* (m integer). Obviously, in the present case, the extended series for each power is (with $v^m(0,t) = 0$, $v^m(L,t) = 0\ \forall m$)

$$v^m(X,t) = \sum_{i=1}^{M} A_{mi}(t)\sin(\frac{i\pi X}{L}) \qquad (m = 1, 2, \cdots) \tag{15}$$

In what follows the expression that relate each $A_{mi}(t)$ with the main unknowns $A_{1i}(t)$ of the problem under study are introduced. In effect, since

$$v^m(X,t) = v^{m-q}(X,t)v^q(X,t) \qquad (m > q \geq 1) \tag{16}$$

and with the following definition (i, j, k integers)

$$P_{ijk} \equiv \frac{1}{L}\int_0^L \sin\frac{i\pi X}{L}\sin\frac{j\pi X}{L}\sin\frac{k\pi X}{L}dX \tag{17}$$

is simply deduced that

$$A_{mi} = 2\sum_{j=1}\sum_{k=1} P_{ijk}A_{(m-q)j}A_{qk} \qquad (m > q \geq 1) \tag{18}$$

The value of $q \geq 1$ is arbitrary and here $q = 1$ was chosen. Now if the following notation is introduced

$$w_1 = \begin{cases} \frac{\gamma_i\left[1-(-1)^{i+j+k}\right]}{\gamma_i^2-(\gamma_j-\gamma_k)^2}, & |j-k| \neq i \\ 0, \quad |j-k| = i \end{cases} \tag{19}$$

$$w_2 = \begin{cases} \frac{\gamma_i\left[1-(-1)^{i+j+k}\right]}{\gamma_i^2-(\gamma_j+\gamma_k)^2}, & j+k \neq i \\ 0, \quad j+k = i \end{cases} \tag{20}$$

Then

$$P_{ijk} = \frac{w_1 - w_2}{2} \tag{21}$$

2.3 Derivation of equation of motion

After the replacement of the above-stated series in the expressions of U, U_R, U_e and T it may be concluded that

$$\mathcal{F} = \int_{t_1}^{t_2} \mathcal{L}dt = \mathcal{F}[A_{1i}] \qquad (i = 1, 2, \cdots) \tag{22}$$

Thus if $a_{1i}(t)$ are admissible directions, the space of admissible directions is given by

$$\mathcal{D}_a(\mathcal{F}) = \mathcal{D}_a[\mathcal{F}(A_{1i}] = \{a_{1i}(t_1) = 0, a_{1i}(t_2) = 0\} \tag{23}$$

and

$$\delta(\cdot) = \frac{d}{d\epsilon}(\cdot)[A_{1i} + \epsilon a_{1i}]\bigg|_{\epsilon=0} \tag{24}$$

E	ρ	R_e	R_i	R_0	L	k_0	C	R
GPa	kg/m^3	m	m	m	m	N/m	kNm	kN
210	7850	0.064	0.054	0.156	2000	10^8	50	200

Table 1: Geometrical and material data for illustration

Thus the first variation of the energy functional writes

$$\delta U = \sum_{i=1} a_{1i} \left\{ \left] \frac{\gamma_i^4}{2L^3} EJ + \gamma A \frac{1-r)}{2} \gamma_i^2 \right] A_{1i} - \right\} \\ \gamma A \gamma_i \sum_{p=1} \gamma_p Y_{ip} A_{ip} \Big\} \quad (25)$$

where

$$Y_{ip} = \begin{cases} \frac{1}{4} & i = p \\ \frac{(\gamma_i^2+\gamma_p^2)((-1)^{i+p}-1)}{(\gamma_i^2-\gamma_j^2)^2}, & i \neq p \end{cases}$$

Also

$$\delta U_R = \frac{k_0 L}{2} \sum_{i=1} a_{1i}(t) \sum_{n=1,3,5,\cdots} G_n A_{ni}(t) \quad (26)$$

$$\delta U_e = \frac{-M(t)}{L} \sum_{i=1} a_{1i}(t)(-1)^i \gamma_i \quad (27)$$

$$\delta T = \frac{\rho A L}{2} \sum_{i=1} \dot{a}_{1i}(t) \dot{A}_{1i}(t) \quad (28)$$

The ordinary differential system that governs the problem is found after the imposition of the stationary of the functional $\mathcal{F}[A_{1i}]$ and integration by parts δT in the time variable keeping in mind that the solution is within the space of admissible directions and that the factors of each a_{1i} $(i = 1, 2, \ldots)$ are set null. Then

$$\rho A L \ddot{A}_{1i} + \gamma_i^2 \left[\frac{EJ\gamma_i^4}{L^3} + (\gamma A)(1-r) \right] A_{1i}(t) - \\ -2\gamma_i \gamma A \sum_{p=1} \gamma_p Y_{ip} A_{ip}(t) + k_0 L \sum_{n=1,3,5,\ldots} G_n A_{ni}(t) = \\ = (-1)^i \gamma_i \frac{M(t)}{L} \quad (29) \\ (i = 1, 2, 3, \ldots)$$

3 EXAMPLE

This system has been solved in this study by means of a standard integration technique (Runge-Kutta) and the results are shown in Figures 4 to 6. An illustration is carried out with the data shown in Table 1. The dynamic moment is $M_0(t) = C \sin 2\pi t$.

The results shown have been found using 60 terms in the spatial series ($M = 60$), i.e. a resulting first order ODE of order 120. The soil response was assumed with $MG = 17$ terms.

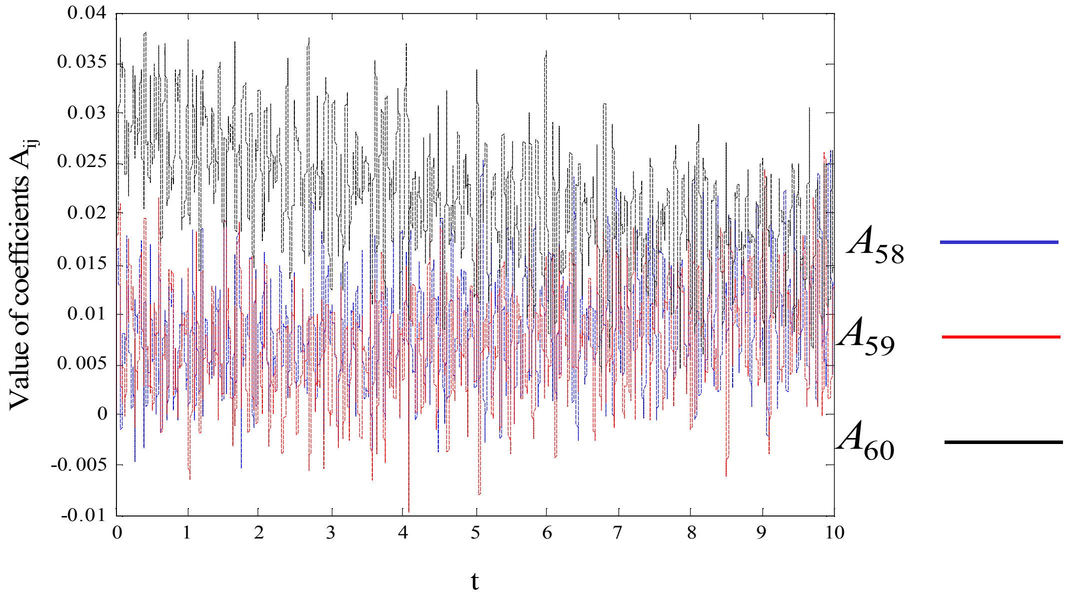

Figure 4: Variation of last coefficients of the spatial series with time. A_{58}, A_{59} and A_{60}.

In Fig. 4 the variation of the last coefficients of the series (A_{58}, A_{59} and A_{60}) with the time is shown. As may be observed, the values are convergent. Also the evolution of the displacement at location $X = 1000$ m ($v(1000, t)$) is depicted in Fig. 5.

The *elastica* at three different intermediate times is included in Fig. 6. The dynamic moment effect is evident at the region close to $X = 2000$ m.

A different approach was used to solve the derived governing equations of the system. The FlexPDE software was employed to find the numerical solution of the governing equation. The space domain was modeld with a finite element grid of 110 cubic elements. By default the software uses Newton time steps. Figure 7 depicts the output for this example. The comparison of the results can be made qualitatively since the latter model has to be adjusted. The contact is modeled in each node of the beam unlike the semi-analytical model herein presented that assumes eventual contact at each point.

4 STATIC PROBLEM

In order to fix ideas the solution of the bar immersed in a discontinuous Winkler medium as above is included in this Section. Here it is subjected to a static moment at its lower end of magnitude M_0; the unknowns are the A_{1i}'s but in this case they are time-independent, i.e.

$$\ddot{A}_{1i} = 0 \qquad (i = 1, 2, 3, \ldots) \tag{30}$$

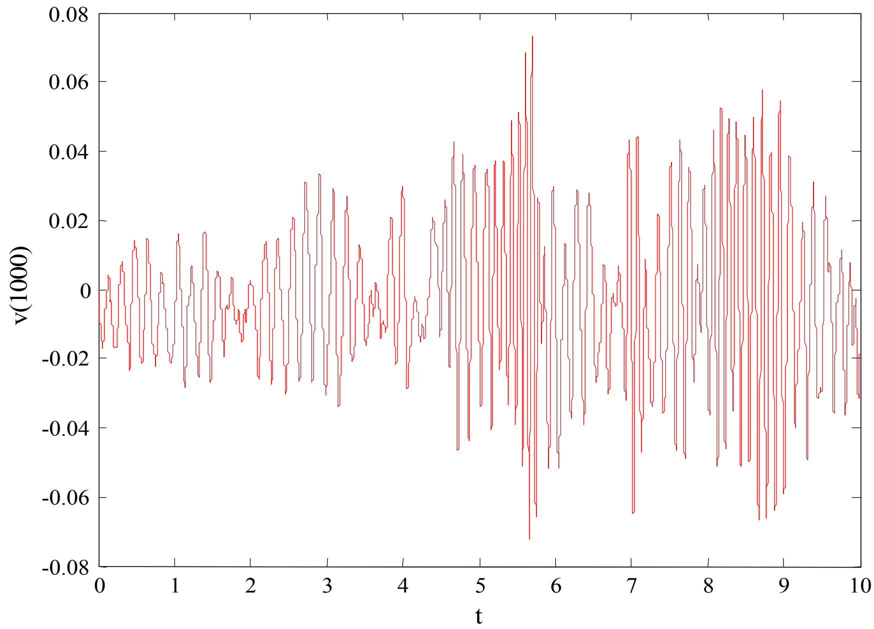

Figure 5: Lateral displacement of the beam at $X = 1000$. Duration of experiment: 10 s.

from which a nonlinear algebraic system yields,

$$\gamma_i^2 \left[\frac{EJ\gamma_i^4}{L^3} + (\gamma A)(1-r) \right] A_{1i}$$

$$-2\gamma_i \gamma A \sum_{p=1} \gamma_p Y_{ip} A_{ip} + k_0 L \sum_{n=1,3,5,\ldots} G_n A_{ni}$$

$$= (-1)^i \gamma_i \frac{M_0}{L} \tag{31}$$

$$(i = 1, 2, 3, \ldots)$$

which is solve by a simple iteration procedure. If M_0 is large enough, the surrounding soil will start to act passively opposing to the deformation until the active and passive loads are statically equilibrated by the response. In Figure 8 the response is shown for different values of r (ratio of the bottom reaction to the rod total weight).

The value $r = 0.4$ appears as a practical limit. The static solution is strongly influenced by the prestress regime assumed for the beam; this is due to a simple stability problem (buckling). The critical load for this problem is presented as an illustration in the Appendix.

5 CONCLUSIONS

The semi-analytical solution of the dynamic behavior of a bar subjected to a prestressed state, located in a borehole with surrounding Winkler soil has been herein presented. The clearance adds a strong nonlinearity to the problem. In a first simplification a Bernoulli-Euler beam of a Hookean material and uniform cross section is subjected to an axial load at the bottom end to simulate the static reaction force when the drill bit touches the formation and, the dead load. Due to the slenderness of the beam the clamped end was assumed as hinged. Such simplification is believed not to modify the global behavior. The energy contribution of the clearance nonlinearity was taken into account by means of a spring and represented by power

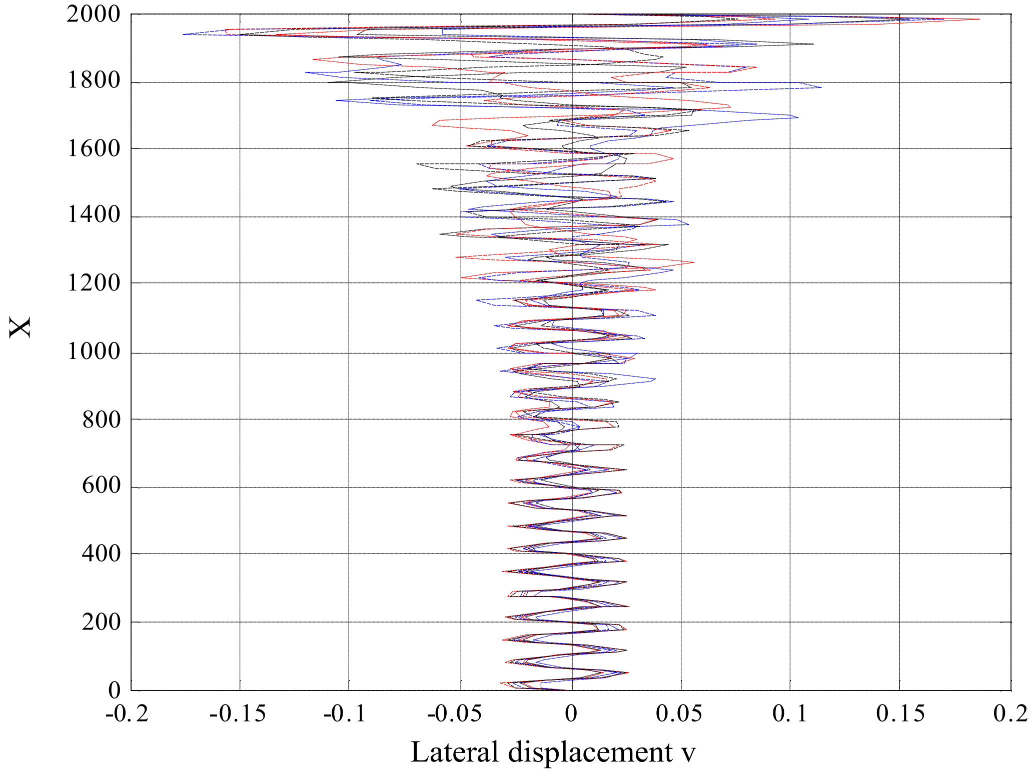

Figure 6: Variation of the *elastica* for three different intermediate times.

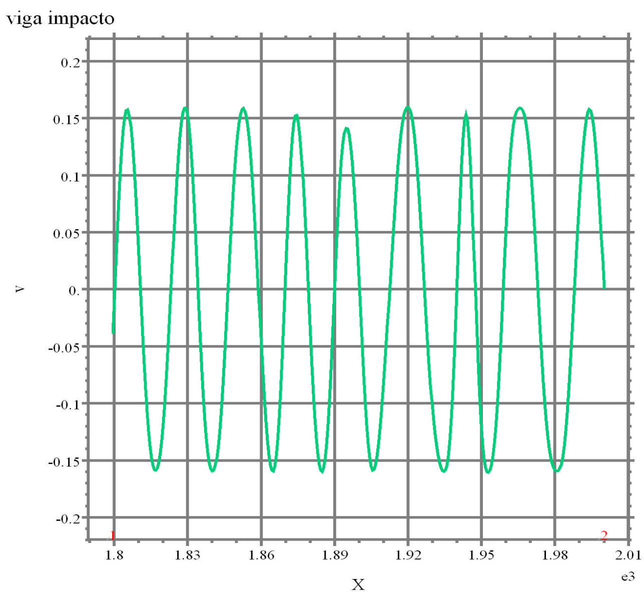

Figure 7: Solution found with finite element discretization of the spatial domain. Lateral displacements at t=10 s along the lower portion of the beam

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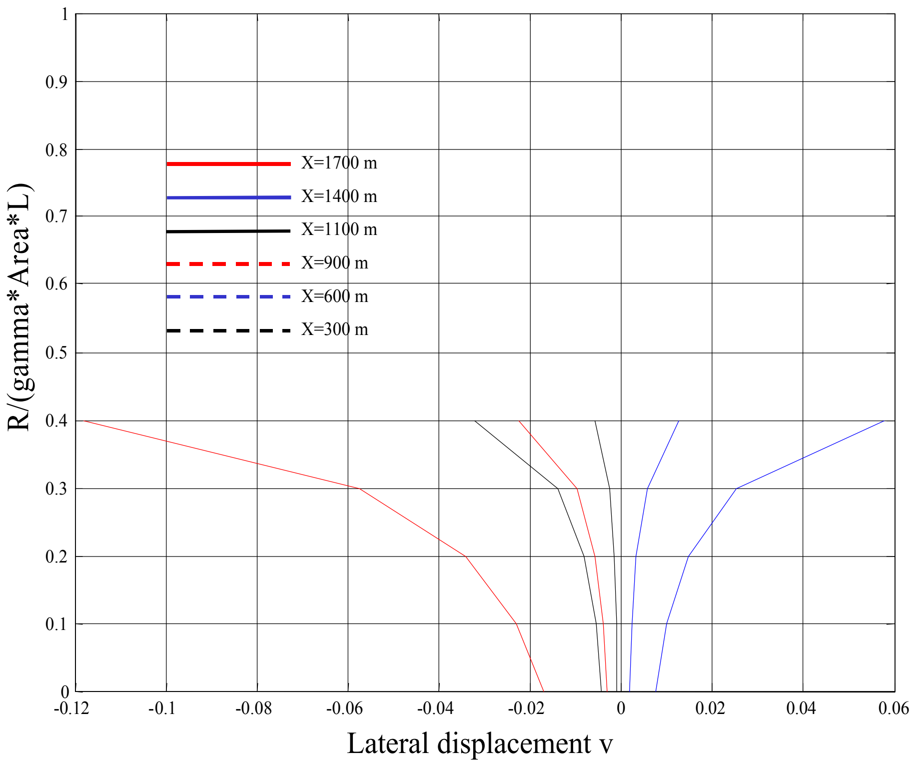

Figure 8: Static displacement at different points of the rod

series expansion in the lateral displacements. The solution was found by first applying a direct method with extended trigonometric series that ensure the uniform convergence of the basic unknown functions in the spacial domain. The resulting nonlinear differential system in the time variable was solved by a standard Runge-Kutta routine within the Matlab environment. The reported results of the deformed configurations are preliminary since the authors continue the study of the numerical convergence. Additionally the static state analysis and the critical load of the prestressed beam were included. Although the beam model was assumed with strong simplifications this is a first step to obtain an semi-analytical solution that may serve as a reference solution to other numerical techniques. The authors are actually analyzing other models considering nonlinear strain energy for the beam and stepped beams. Some of this analyzes are expected to be reported at the ENIEF 2006.

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Table 2: Critical loads for a simply supported column with variable axial stress. c

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$P_{cr}\left(\frac{AL^2}{EJ}\right)$	18.6	22.5	28.4	37.7	53.6	83.2	144.2	296.5

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A CRITICAL LOAD OF A SIMPLY SUPPORTED ROD WITH LINEARLY VARYING NORMAL FORCE

Let us analyze the equilibrated straight configuration under the stress

$$\sigma_I(X) = \gamma L(c - \frac{X}{L}); \qquad (0 \leq X \leq L) \tag{32}$$

where $c = 1 - r$. The total load is denoted $P = \gamma AL$. In order to impose other equilibrated (bending type-Euler) configuration (II) in the vicinity to (I) measured by $v = v(x)$, that is obtained under the critical load magnitude $P = P_{cr} = \gamma AL_{cr}$, the following differential equation has to be satisfied

$$v'''' + P_{cr}(\frac{AL^2}{EJ})[v' - (c - x)v''] = 0 \tag{33}$$

where $x = X/L$ $(0 \leq x \leq 1)$, $(\cdot)' \equiv d(\dot{})/dx$, E is the Young's modulus,A and J are the area and the moment of inertia of the uniform section of the rod, respectively. The problem is solved for the simply supported bar by means of a power series solution, yielding the values reported in Table 2. It was concluded that $M \geq 40$ fulfills the convergence of the results.

It should be noted that for $c \geq 0.79$ (prevailing tensile state) no critical load is found.