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DISCONTINUOUS GALERKIN METHOD FOR THE ONE DIMENSIONAL SIMULATION OF SHALLOW WATER FLOWS

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Abstract. *A numerical solution for the one-dimensional (1D) hyperbolic conservation law is presented, based on the Runge Kutta Discontinuous Galerkin Method (RKDG). The RKDG scheme combines some properties of the finite element and finite-volume techniques, resulting on a very attractive method because of its formal high-order accuracy, its ability to handle complicated geometries, its adaptability to parallelization, and its ability to capture discontinuities without producing spurious oscillations. In this paper, we consider some scalar conservation equations to ilustrate the method's properties in one spatial dimension (1-D). Finally, the 1-D shallow water equations are discretized with the RKDG. A comparison with an exact solution is made to illustrate the capability of the method to handle strong discontinuities with relative small number of elements.*

1 INTRODUCTION

The Runge Kutta Discontinuous Galerkin (RKDG) method has been introduced in the 90's with a series of papers by Cockburn and Shu[1–4] as an alternative to the so-called Discontinuous Galerkin (DG) method. The Discontinuous Galerkin (DG) method was originally introduced by Reed and Hill[5] for solving linear neutron transport equations, and studied by Jamet[6] and Jonhson and Pitkaranta,[7] among others. Whereas in the DG method, the approximate solution is discontinuous only in time, with continuos finite element in space dimensions, in the RKDG the solution is approximated in space with discontinuous -or piecewise continuous- functions. Chavent and Salzano[8] constructed an explicit version of the DG for the one-dimensional scalar conservation law. They discretized the equation by using piecewise linear elements in space and by using the Euler forward step method in time. The scheme was unconditionally unstable even for very restrictive conditions on the time step and mesh size. To improve the stability of the method, Chavent and Cockburn[9] modified the scheme by introducing a *slope limiter* borrowing the ideas introduced by van Leer[10] . The scheme so obtained was proven to be total variation diminishing in the means (TVDM) and total variation bounded (TVB). However, the scheme was only first order accurate in time and the adopted *slope limiter* had to damp oscillations triggered by linear instability even in smooth regions of the solution, thus degrading the quality of the results. These dificulties were overcome by Cockburn and Shu,[4] where the first RKDG method was introduced with the following attributes: (i) the method retained the piecewise linear approximation of the DG method in space, (ii) the method used an explicit TVD second order accurate Runge-Kutta discretization developed by Shu and Osher[11] for finite difference schemes. Then, Cockburn and Shu[2] extended the approach to devise RKDG of higher orders. Consequentley, the RKDG method has found rapid applications in such diverse areas of gas dynamics,[12] transport of contaminant in porous media,[13] and shallow water flows[14,15] among many others problems of the water and the environment. See, for example, the review paper of Cockburn for an extended bibliography and applications[16] .

The RKDG method has several advantages that well-deserved consideration: (*i*) it preserves the well-known capability of the classical finite element method to handle complicated geometries, (*ii*) it handles adaptivity strategies very easily, since the grid refinement can be done without taking into account the continuity requirement typical of conforming finite element methods. (*iii*) it increases the degree of the aproximating polynomials locally, thus allowing an efficient p adaptivity for each element with total independence of its neighbors. (*iv*) it comunicates each element data with its inmediate neighbors only, regardless of the order of accuracy, thus allowing for efficient parallel implementations,[17] and (*v*) it has very good stability properties without the need for limiters in many situations[11,13] .

In this paper, the scalar hyperbolic equation is considered first to ilustrate the method's properties in one spatial dimension (1-D). Then, the 1-D linear diffusion is analyzed, followed by a brief discussion of 1-D advection-diffusion applications. Finally, the 1-D shallow water equations are discretized with the RKDG. A comparison with an exact solution is made to illustrate the capability of the method to handle strong discontinuities with relative small number of elements.

2 THE SCALAR HYPERBOLIC CONSERVATION LAW IN ONE SPACE DIMENSION

The basic idea of the method can be ilustrated with the scalar hyperbolic conservation law in 1-D

$$u_t + f(u)_x = 0 \quad , \quad (0,1)\times(0,T) \tag{1}$$
$$u(x,0) = u_o(x) \tag{2}$$

subjet to periodic boundary conditions. If the mesh partition of the domain $(0,1)$ is denoted by $\{I_j\}_{j=1}^N$, where the grid size is $\Delta_j = x_{j+1/2} - x_{j-1/2}$, and the center of the element is denoted by $x_j = (x_{j-1/2} + x_{j+1/2})/2$, the weak statement of the problem is obtained if (1) and (2) are multiply by an arbitrary smooth function $v(x)$, and integrated by parts over the interval I_j.

$$\int_{I_j} \partial_t u\, v dx = \int_{I_j} f(u)v' dx + f[u(x^+_{j-\frac{1}{2}},t)]\, v(x^+_{j-\frac{1}{2}}) - f[u(x^-_{j+\frac{1}{2}},t)]\, v(x^-_{j+\frac{1}{2}}) \tag{3}$$

$$\int_{I_j} u(x,0)v(x)dx = \int_{I_j} u_o(x)v(x)dx \tag{4}$$

where $x^-_{j+1/2}$ denotes the limit from the left, and $x^+_{j-1/2}$ the limit from the right. Then, the variational statement of the problem is this: find an approximate solution u_h to u for each time $t \in (0,T)$,where $u_h(x,t)$ and $v(x)$ belongs to the finite dimensional space

$$V_h^K = \left\{ v \in L^1(0,1) \;:\; v|_{I_j} \in P^K(I_j),\; j = 1,...,N \right\} \tag{5}$$

That is, in the space dimension V_h^K, u_h and v are piecewise polynomials of degree at most K. Since both functions u_h and v are discontinuos at the points $x_{j\pm1/2}$, the ambiguity present in the last two terms of (3) involving the non-linear fluxes $f[u(x_{j\pm1/2},t)]$ must be replaced by *numerical fluxes* that depend on the values of u_h at the interfaces $x_{j\pm1/2}$

$$\widehat{f}_{j+1/2} = \widehat{f}(u_h|^-_{j+1/2}\, , u_h|^+_{j+1/2}) \quad , \quad \widehat{f}_{j-1/2} = \widehat{f}(u_h|^-_{j-1/2}\, , u_h|^+_{j-1/2}) \tag{6}$$

that will be suitably chosen later (the idea is to define these numerical fluxes by an upwinding mechanism, i.e., with information defined along characteristics). Thus, if the approximate solution u_h is expanded in terms of the Legendre's polynomials P_m as local basis functions

$$u_h(x,t)|_{I_j} = \sum_{m=0}^{K} u_{mj}(t) P_m\left(\frac{2(x-x_j)}{\Delta_j}\right) \quad , \quad \forall\, j = 1, ..., N \quad , \tag{7}$$

the test functions $v(x)$ are taken to be the same as the basis functions, i.e., $v(x) = \{P_l\}_{l=0}^{K}$, which is the esence of the Bubnov-Galerkin method, and the orthogonality property of the Legendre's polynomials is invoked

$$\int_{-1}^{1} P_m(\xi) P_l(\xi) d\xi = \frac{2}{2l+1}\,\delta_{ml} \tag{8}$$

where

$$\xi = \frac{2(x-x_j)}{\Delta_j} \quad , \quad \delta_{ml} = \begin{cases} 1\ , & m = l \\ 0\ , & otherwise \end{cases} \quad , \tag{9}$$

the weak form simplify to

$$\frac{du_{lj}(t)}{dt} = \frac{2l+1}{\Delta_j} \int_{-1}^{1} f[u_h(\xi,t)]\, P_l'(\xi)\, d\xi + \frac{2l+1}{\Delta_j} \left\{(-1)^l\, \widehat{f}^{+}_{j-\frac{1}{2}} - \widehat{f}^{-}_{j+\frac{1}{2}}\right\} \tag{10}$$

$$u_{lj}(0) = \frac{2l+1}{2} \int_{-1}^{1} u_o(\xi) P_l(\xi) d\xi \quad , \quad \forall l = 0, ..., K, \;\; and \;\; j = 1, ..., N \tag{11}$$

where the properties of the Legendre's polynomials $P_l(-1) = (-1)^l$, $P_l(1) = 1$ have been used.

2.1 Numerical Fluxes

To complete the definition of the problem, it remains to choose the numerical flux. Here, it is cruxial to stress the point made by Cockburn[16] '... the idea is to construct schemes that are *perturbations* of the so-called monotone schemes. That is, by *perturbing* the monotone scheme, it is possible to achieve high order accuracy while keeping their stabilitiy and convergence properties...'. Consequently, when the DG-space discretization is piecewise constant, $K = 0$, the integration scheme must be monotone

$$\frac{du_{oj}(t)}{dt} = \frac{\widehat{f}_{j+\frac{1}{2}} - \widehat{f}_{j-\frac{1}{2}}}{2} \quad , \quad \forall j = 1, ..., N \tag{12}$$

$$u_{oj}(0) = \frac{1}{2} \int_{-1}^{1} u_o(\xi)\, d\xi \tag{13}$$

This defines a monotone scheme if $\widehat{f}(a,b)$ is a Lipschitz continuous, consistent, and monotone flux. That is, if it is (i) locally Lipschitz, and consistent with the flux $f(u)$, i.e., $\widehat{f}(u,u) = f(u)$, (ii) a nondecreasing function of its first argument, and (iii) a nonincreasing function of its second argument. There are several examples of numerical fluxes that satisfies the above properties. Here, the local Lax-Friedrichs flux $\widehat{f}^{LLF}$ are considered

$$\widehat{f}^{LLF}(a,b) = \begin{cases} \frac{1}{2}\left[f(a) + f(b) - C(b-a)\right] \\ C = \max|f'(s)|, \quad \min(a,b) \le s \le \max(a,b) \end{cases} \tag{14}$$

2.2 The TVD-Runge-Kutta algorithm for time integration

The discretized system, once the numerical fluxes are specified, can be written as follows

$$\begin{aligned} \frac{du_h(t)}{dt} &= L_h(u_h,t) \quad , \quad \forall t \in (0,T) \\ u_h(0) &= u_o^h \end{aligned}$$

The elements of $L_h(u_h)$ of V_h^K are the outcome of approximating $-f(u(x,t))_x$ by the DG-space discretization. When polynomial of degree K are used, a Runge-Kutta method of order $(K+1)$ must be employed.[16] Then, if the TVD Runge-Kutta time discretization introduced in Shu[18] is used, the time-stepping algorithm reads as follows: if $\{t^n\}_{n=0}^{M}$ is a partition of $[0,T]$ in M time intervals, and $\Delta t^n = t^{n+1} - t^n$, for $n = 0, ..., M.$, the simplest TVD Runge-Kutta of orden two is given by

$$u_h^{(1)} = u_h^n + \Delta t\, L_h(u_h^n) \tag{15}$$

$$u_h^{n+1} = \frac{1}{2}\,u_h^n + \frac{1}{2}\left\{\left[u_h^n + \Delta t\, L_h(u_h^n)\right] + \Delta t\, L_h(u_h^{(1)})\right\} \tag{16}$$

2.3 Remarks on the stability of the method

In general, it is possible to obtain the stability of the method from the analysis of a single "Euler forward" step. However, the stability of the complete method must be considered at once. Cockburn[16] cites the following stability limit for the linear flux $f(u) = cu$ (c constant) for $K = 2$

$$c\frac{\Delta t}{\Delta x} < \frac{1}{3} \quad for \;\; K=1, \quad c\frac{\Delta t}{\Delta x} < \frac{1}{5} \quad for \;\; K=2 \tag{17}$$

2.4 The TVDM generalized slope limiter

When high order polynomials are used to approximate the solution, the higher order terms must be controlled to inhibit oscillations. Cockburn and Shu[2] developed an explicit third order TVD Runge-Kutta procedure in time combined with local projection operators in space to control numerical oscillations. Thus, if v_h represents the linear part, or projection of u_h

$$v_h = u_{o,j} + \frac{2(x-x_j)}{\Delta_j}\,u_{1,j} \tag{18}$$

the slope limiter of van Leer that reads

$$v_h\big|^-_{j+\frac{1}{2}} = u_{o,j} + m_m(u_{1,j},\, u_{o,j+1} - u_{o,j},\, u_{o,j} - u_{o,j-1}) \tag{19}$$

$$v_h\big|^+_{j-\frac{1}{2}} = u_{o,j} - m_m(u_{1,j},\, u_{o,j+1} - u_{o,j},\, u_{o,j} - u_{o,j-1}), \tag{20}$$

limit the amplification of the solution slope, where m_m is the minmod function defined as follows

$$m_m(a_1,...,a_\nu) = \begin{cases} s\ \min_{1\leq n\leq \nu} |a_n| \ , & if\ \ s = sign(a1) = ... = sign(a_\nu) \\ 0 \ , & otherwise \end{cases} \tag{21}$$

Shu[19] modified this slope limiter to preserve high-order accuracy at local extrema of the function. The resulting scheme turned out not to be TVDM but total variation bounded in the means (TVBM). For details, see Cockburn[16] .

3 THE DIFFUSSION EQUATION

The original idea that supports the RKDG method discussed above can be extended very easily to difussion, or advection-difussion dominated problems. The idea can be ilustrated with the simple heat equation

$$u_t - u_{xx} = 0 \tag{22}$$

which can be rewriten as a first order system

$$u_t - q_x = 0 \tag{23}$$

$$q - u_x = 0 \tag{24}$$

It is possible to use *formally* the same RKDG method discussed above for hyperbolic problems, resulting in the following variational statement: find $u_h,\, q_h \in V_h^K$ such that, for all test functions $v_1,\, v_2 \in V_h^K$

$$\int_{I_j} v_1 \partial_t u_h\, dx = -\int_{I_j} q_h\, \partial_x v_1 dx + \widehat{q}_{j+\frac{1}{2}}\, v_1\Big|^-_{j+\frac{1}{2}} - \widehat{q}_{j-\frac{1}{2}} v_1\Big|^+_{j-\frac{1}{2}} \tag{25}$$

$$\int_{I_j} v_2\, q_h\, dx = -\int_{I_j} u_h\, \partial_x v_2 dx + \widehat{u}_{j-\frac{1}{2}}\, v_2\Big|^-_{j+\frac{1}{2}} - \widehat{u}_{j-\frac{1}{2}}\, v_2\Big|^+_{j-\frac{1}{2}} \tag{26}$$

However, in this case there is no upwinding mechanisms or characterist directions to guide the design of the numerical fluxes $\widehat{u}$, $\widehat{q}$. Indeed, the crucial part of designing a stable and accurate algorithm to solve the above equations is to design proper numerical fluxes. The most natural fluxes seem to be the central average

$$\widehat{u}_{j+\frac{1}{2}} = \frac{1}{2}\left(u^+_{j-\frac{1}{2}} + u^-_{j+\frac{1}{2}}\right) \ , \ \ \widehat{q}_{j+\frac{1}{2}} = \frac{1}{2}\left(q^+_{j-\frac{1}{2}} + q^-_{j+\frac{1}{2}}\right)$$

The appearance of the auxiliar variable q is superficial; when a local basis is chosen within element I_j, q is afterwards trivially eliminated and the resultant scheme for u is similar to that deviced for hyperbolic problems. Last but not least, both schemes developed by the scalar hyperbolic problem and the difussion problem in 1-D can be combined to gives rise to the so-called Local Discontinuous Galerkin for solving linear and no-linear advection-diffusion problems[16] .

4 COMPUTATIONAL RESULTS

The simple *scalar wave equation* with periodic boundary conditions is considered first

$$u_t + u_x = 0, \qquad u_o(x) = \begin{cases} 1 \ , & 0.4 \leq x \leq 0.6 \\ 0 \ , & otherwise \end{cases} \tag{27}$$

To magnify the effect of the dissipation of the method, the solution is advected 100 times ($T = 100$) across the periodic domain. The scheme is let run with a CFL (Courant-Friedrichs-Lewy condition) equal to $0.9 \times 1/5 = 0.18$ for $K = 0, 1$, and 2. It can be seen in Figure 1 that, for $N = 80$ elements, the dissipation effect decreases as the degree of the approximating polynomial increases.

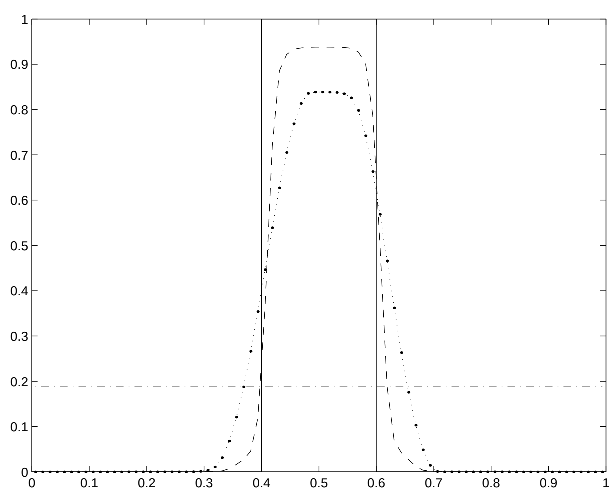

Figure 1: The simple scalar wave equation, $T = 100$, $N = 80$. Exact solution (solid line), $K = 0$ (dash/dotted line), $K = 1$ (dotted line), and $K = 2$ (dashed line).

For the *difussion equation*, the following problem is adopted with periodic boundary conditions

$$u_t - u_{xx} = 0 \quad , \quad u_o(x) = \begin{cases} 10x - 4 \ , & 0.4 \leq x \leq 0.5 \\ -10x + 6, & 0.5 < x \leq 0.6 \\ 0 \ , & otherwise \end{cases}$$

on the domain $x \in [0, 1]$. Figure 2 shows the evolution of the solution when the domain is partitioned in $N = 40$ elements. The solution shows how the auxiliary variable, $q_h = \partial_x u_h$, approximates the derivative of the solution with the same order of accuracy as u, thus matching traditional advantages of finite element methods based on 'equal order of interpolation' techniques.

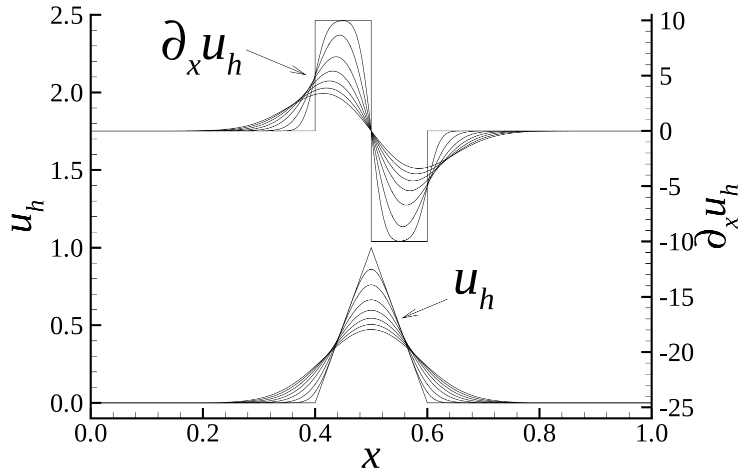

Figure 2: The diffusion equation.

The one dimensional, *linear convection diffusion equation*

$$u_t + c\,u_x - a\,u_{xx} = 0 \quad \text{in}\,(0,T)\times(0,2\pi), \tag{28}$$

where c and $a \geq 0$ are both constants, is tested with the initial condition $u(t = 0, x) = sin(x)$ and periodic boundary conditions. The exact solution is $u(x,t) = e^{-at}\sin(x - ct)$. The solution is computed up to $T = 2$, and a comparison between the exact solution and numerical solution is shown in Figure 3.

The one dimensional, *non linear convection diffusion equation (Burgers equation)* is given by the following equation

$$u_t + (\frac{u^2}{2})_x - a\,u_{xx} = 0, \quad \text{in}\,(0,T)\times(0,2), \tag{29}$$

$$u(t=0,x) = \sin(\pi\,x), \tag{30}$$

subject to the homogeneous boundary conditions $u(0) = u(2) = 0$. This equation was originally introduced by J. M. Burgers and represents a simplified model of the more complicated Navier-Stokes equations. Figure 4 shows the evolution of a sinusoidal wave

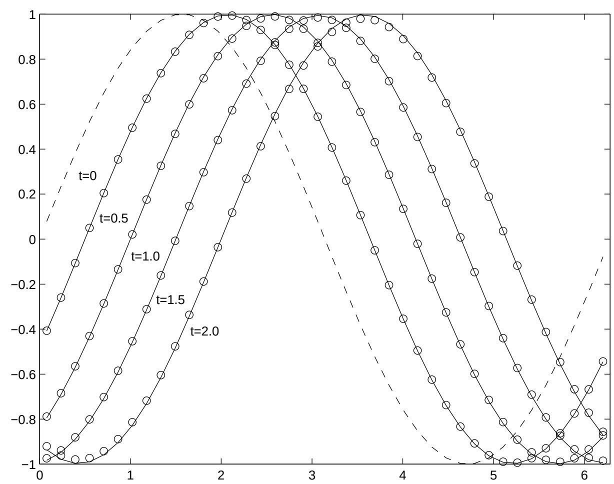

Figure 3: Linear convection diffusion equation: initial condition $T = 0$ (dashed line), exact solution (circle), numerical solution (solid line) at $T = 0.5$, 1.0, 1.5, and 2.0, $c = 1$, $a = 0.01$.

governed by the viscous Burgers equation with $a = 10^{-2}/\pi$, using $N = 80$ elements. The structure of the wave is shown from $T = 0$ to $T = 2$ and it can be seen that the shock is very-well captured without the oscillations and the overshoots that hamper the quality of the solutions of many other methods.

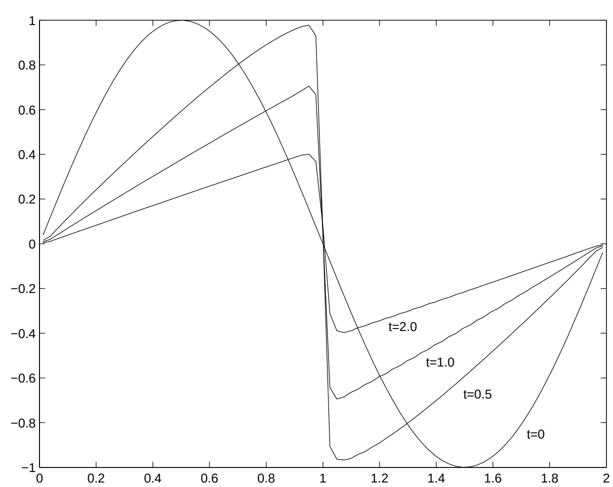

Figure 4: Non linear convection diffusion equation, $a = 10^{-2}/\pi$.

5 THE ONE-DIMENSIONAL SHALLOW WATER EQUATIONS

The aforemencioned ideas and algorithms of the RKDG method devised to solve hyperbolic and advection-diffusion problems carry over almost intact to more complicated hyperbolic problems: in this case, the Shallow Water Equations (SWE).

5.1 Governing Equations

The one-dimensional SWE, in general conservative form are given by

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}), \quad \text{in } (0,L)\times(0,T) \tag{31}$$

where $\mathbf{U}$ is the vector of conserved variables, $\mathbf{F}$ is the flux vector in the x direction, $\mathbf{S}$ represent a source vector, and t is the time. The vectors $\mathbf{U}$ and $\mathbf{F}$ are expressed as

$$\mathbf{U} = \left[\begin{array}{c} h \\ q \end{array}\right], \quad \mathbf{F} = \left[\begin{array}{c} q \\ q^2/h + gh^2/2 \end{array}\right], \tag{32}$$

where g is the acceleration due to gravity, q is the dischage per unit width $= uh$, h the water depth, and u is the flow velocity in the $x-$direction.
The source term $\mathbf{S}$ is given by

$$\mathbf{S} = \left[\begin{array}{c} 0 \\ gh(S_0 - S_f) \end{array}\right], \tag{33}$$

which contains the effects of the bed slope S_0 and the bed friction S_f. The term S_f can be estimated by an empirical formulae, for example by the Manning resistance law

$$S_f = \frac{n^2 q|q|}{h^{10/3}}, \tag{34}$$

in which n is the Manning resistance coefficient.

5.2 The DG space discretization

Following Cockburn[16] , for each partition of the interval $(0,L)$, $\{x_{j+1/2}\}_{j=0}^{N}$ we set $I_j = (x_{j+1/2}; x_{j-1/2})$, and $\Delta_j = x_{j+1/2} - x_{j-1/2}$ for $j = 1,\cdots,N$. Then, an approximation $\mathbf{U}_h = (h_h, q_h)^T$ to $\mathbf{U}$ is sought that for each time $t \in [0,T]$, $\mathbf{U}_h(t)$ belongs to the finite dimensional space of polynomials $P^K(I)$ in I of degree at most K. In order to determine the approximate solution $\mathbf{U}_h$, a weak formulation of the problem is obtained by multiplying the equation (31) with an arbitrary, smooth function v and integrating over I_j by parts

$$\begin{aligned} \int_{I_j} \partial_t \mathbf{U}(x,t)\; v(x)dx \; - \; \int_{I_j} \mathbf{F}(\mathbf{U})(x,t)\partial_x \; v(x)dx \\ +\mathbf{F}(\mathbf{U})(x_{j+1/2},t)\, v(x^-_{j+1/2}) \; - \; \mathbf{F}(\mathbf{U})(x_{j-1/2},t)\, v(x^+_{j-1/2}) \\ = \int_{I_j} \mathbf{S}(\mathbf{U})(x,t) v(x)dx \end{aligned} \tag{35}$$

Next, the exact solution $\mathbf{U}$ is replaced by the approximate solution $\mathbf{U}_h$ belonging to V_h^K.
Note that the function $\mathbf{U}_h$ is discontinuous at the points $x_{j+1/2}$, so it is necessary to replace the nonlinear flux $\mathbf{F}$ by a *numerical* flux $\mathbf{H}$ that depends on the two values of $\mathbf{U}_h$ at the point $(x_{j+1/2}, t)$, that is, by the function

$$\mathbf{H}(\mathbf{U})_{j+1/2}(t) = \mathbf{H}(\mathbf{U}(x^-_{j+1/2}, t), \mathbf{U}(x^+_{j+1/2}, t)) \tag{36}$$

are discussed later. Thus, the appropiate solution given by the DG space discretization for the one-dimensional SWE is defined as the solution of the following weak formulation:

$$\begin{aligned} &\forall\ j = 1, \cdots, N, \quad v \in P^k(I_j) : \\ &\int_{I_j} \partial_t \mathbf{U}_h(x,t)\ v(x)dx \ -\ \int_{I_j} \mathbf{F}(\mathbf{U}_h)(x,t)\partial_x\ v(x)dx \\ &+\mathbf{H}(\mathbf{U}_h)_{j+1/2}(t)\, v(x^-_{j+1/2}) \ -\ \mathbf{H}(\mathbf{U}_h)_{j-1/2}(t)\, v(x^+_{j-1/2}) \\ &= \int_{I_j} \mathbf{S}(\mathbf{U}_h)(x,t) v(x) dx \end{aligned} \tag{37}$$

If the Legendre polynomials are chosen for the finite element basis V_h^K. The approximate solution $\mathbf{U}_h$ is then expanded as

$$\mathbf{U}_h(x,t) = \sum_{l=0}^{K} \mathbf{U}_j^l(t) P_l(x), \tag{38}$$

where the Legendre polynomials are given by

$$\begin{aligned} P_1(x) &= 1. \\ P_2(x) &= 2(x - x_j)/\Delta_j \\ P_3(x) &= \frac{3(x - x_j)^2 - 1}{2} \end{aligned}$$

Finally, the weak formulation (37) takes the form:

$$\begin{aligned} &\forall\ j = 1, \cdots, N,\ \text{and}\ l = 0, \cdots, k \\ &\left(\frac{1}{2l+1}\right) \partial_t \mathbf{U}_j^l \ -\ \int_{I_j} \mathbf{F}(\mathbf{U}_h)(x,t) P_l'(x) dx \\ &+\frac{1}{\Delta_j} \left\{ \mathbf{H}(\mathbf{U}_h)_{j+1/2}(t) \ -(-1)^l\ \mathbf{H}(\mathbf{U}_h)_{j-1/2}(t) \right\} \\ &= \int_{I_j} \mathbf{S}(\mathbf{U}_h)(x,t) P_l(x) dx \end{aligned} \tag{39}$$

5.3 The numerical flux

To complete the discretization in space, it remains to choose the numerical flux $\mathbf{H}$. The DG scheme will be monotone if $\mathbf{H}(\mathbf{U}_L), \mathbf{H}(\mathbf{U}_R))$ is a locally Lipschitz, consistent, and monotone flux. For the SWE, Fraccarollo and Toro[20] presented the HLL (Harten-Lax-van Leer) scheme, based on the work of Harten *et* al.[21] This scheme takes into account the left and right characteristics, and results in three states that are separated by two characteristics

$$\mathbf{H}^{\mathrm{HLL}}(\mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{F}_L & \text{if} \quad \lambda_{\min} \geq 0 \\ \mathbf{F}^* & \text{if} \quad \lambda_{\min} < 0 \ \lambda_{\max} > 0 \\ \mathbf{F}_R & \text{if} \quad \lambda_{\max} \leq 0 \end{cases} \quad (40)$$

where $\mathbf{F}_L = \mathbf{F}(\mathbf{U}_L)$, $\mathbf{F}_R = \mathbf{F}(\mathbf{U}_R)$, and $\mathbf{F}^*$ are given by the equation

$$\mathbf{F}^* = \frac{\lambda_{\max}\mathbf{F}_L - \lambda_{\min}\mathbf{F}_R + \lambda_{\max}\lambda_{\min}(\mathbf{U}_R - \mathbf{U}_L)}{\lambda_{\max} - \lambda_{\min}} \quad (41)$$

The wave speeds are chosen under assumption of two-rarefaction waves,

$$\begin{aligned} \lambda_{\min} &= \min(u_L - \sqrt{g\,h_L}, u^* - \sqrt{g\,h^*}) \\ \lambda_{\max} &= \max(u_R - \sqrt{g\,h_R}, u^* + \sqrt{g\,h^*}), \end{aligned} \quad (42)$$

which

$$\begin{aligned} u^* &= \frac{u_L + u_R}{2} + \sqrt{g\,h_L} - \sqrt{g\,h_R} \\ \sqrt{g\,h^*} &= \frac{u_L - u_R}{4} + \frac{(\sqrt{g\,h_L} + \sqrt{g\,h_R})}{2} \end{aligned} \quad (43)$$

The expressions for the wave speeds were obtained assuming wet bed. For the right dry bed problem, these speeds are[20]

$$\begin{aligned} \lambda_{\min} &= u_L - \sqrt{g\,h_L} \\ \lambda_{\max} &= u_L + 2\sqrt{g\,h_L} \end{aligned} \quad (44)$$

5.4 The TVD Runge-Kutta time discretization for the SWE

Once the system has been discretized in space using the DG method, the system is integrated forward in time using explicit Runge-Kutta procedures as described before. When using constant approximations, first order Runge-Kutta method (explicit Euler) is used. When using linear approximations, second order Runge-Kutta method is used for temporal discretization, and for quadratic approximations, a third order Runge-Kutta method is used. In order to prevent non-physical oscillations, for the space discretizations $K \geq 1$, a slope limiter is applied on every result of the Runge-Kutta method.

5.5 Hypotetical example: Dam break problem

The analytical solution of this problem is given in Stoker[22] . This test consider a wide channel having a barrier placed across its width, where h_1 and h_2 are the water depth upstream and downstream, respectively. At time $t = 0$, the barrier is suddenly removed. The flow consists of a bore travelling downstream and a rarefaction wave travelling upstream. A long channel with zero friction and zero bed slope is then considered for testing the RKDG scheme. A dam at position $x = 0.5$ divides the channel in an upstream and a downstream section. The initial conditions of the problem are given by

$$\mathbf{U}(x,0) = \begin{cases} (h_1, 0)^T & \text{if } x \leq 0.50 \\ (h_2, 0)^T & \text{if } x > 0.50 \end{cases}$$

The performance of the RKDG scheme is showed in Figure 5. In this test, the space discretization has a resolution of 10 elements with constant, linear, and quadratic elements. For $K = 1, 2$ the TVBM slope limiter with $M = 50$ is used. As it can be seen, the shock is captured within only two elements. Figure 6 shows a comparison between the exact solution and the corresponding numerical solution with different partition of the domain ($N = 100$ and $N = 1000$) at time $T = 0.1$. As can be seen, the solution is correctly obtained and the shock is captured with sharp profile in both cases.

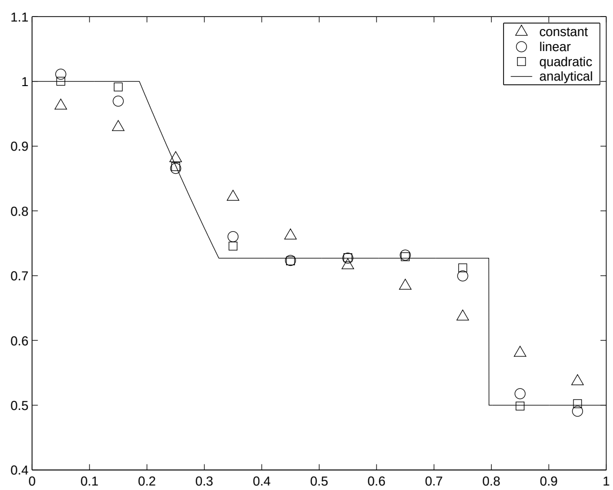

Figure 5: Water depth for the one-dimensional dam-break problem at $T = 0.1$ with $K = 0, 1, 2$, $M = 50$ and $N = 10$: exact solution (solid line), piecewise constant solution (triangle), piecewise linear solution (circle), and piecewise quadratic solution (square).

6 CONCLUSIONS

In this work, a high-order RKDG finite element scheme is proposed for the numerical solution of the one-dimensional (1D) hyperbolic conservation law. The RKDG scheme

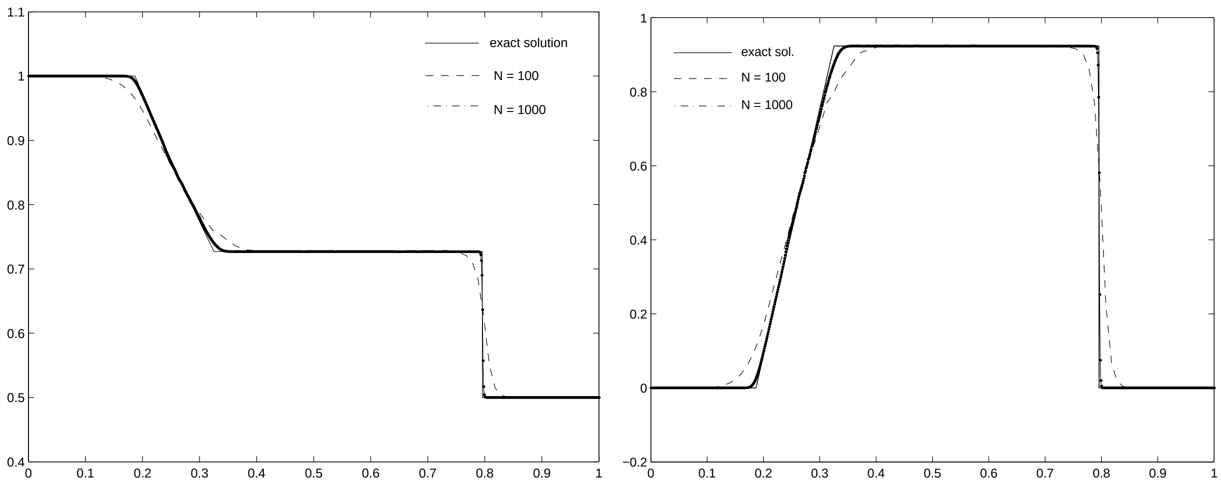

Figure 6: Water depth and velocity for $h_1 = 1.$ and $h_2 = 0.5$ at time $T = 0.1$.

combines some properties of the finite element and finite-volume techniques, resulting on a very attractive method because of its formal high-order accuracy, its ability to handle complicated geometries, its adaptability to parallelization, and its ability to capture discontinuities without producing spurious oscillations. The good agreement between simulated results with analytical solutions shows the ability of the method to capture the shock fronts. Mobile bed extension will be further developments of the model.

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