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EIGENVALUE AND SINGULAR VALUE OPTIMIZATION

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Abstract: *Eigenvalues, singular values and condition number of matrices, play an important role in many fields of applied mathematics to engineering. Among the plethora of applications of eigenvalues in mathematics and engineering we can mention numerical analysis, structural design, quantum mechanics and system dynamics (physical and biological models). For some applications it may be desirable to choose the parameters of a model in order to optimize an objective function and/or to verify constraints that involve eigenvalues or singular values of a certain matrix. In general, the elements of the matrix depend in a nonlinear fashion on the optimization parameters. The purpose of this contribution is to introduce recent formulations of eigenvalue and singular value optimization as well as techniques to include condition numbers within the optimization problem. A chemical engineering design problem is presented to illustrate the proposed techniques.*

1. INTRODUCTION

Among the plethora of applications of eigenvalues in mathematics and engineering, it can be mentioned numerical analysis, structural design, quantum mechanics and system dynamics (physical, chemical and biological models). Singular values and condition number of matrices are also defined in terms of eigenvalues.

In problems involving eigenvalues, in general it is not just the case to calculate the eigenvalues of a given matrix **A**. It is more typical that the elements of matrix **A** depend on an amount of variables, say **y**, and that the values of such variables are desired to be the solution of some optimization problem involving the eigenvalues of **A**(**y**) as objective functions and/or constraints. For a comprehensive survey on the subject, which also includes an historical account of the development of the field, see Lewis and Overton[1].

Few contributions dealing with the general unsymmetric, non-linear case have been presented. Much of the work in nonlinear eigenvalue optimization, from both theoretical and algorithmic points of view, has been produced in the mechanical (structural) engineering field[2].

It is not the aim of this paper to make a review of the field of eigenvalue optimization but to introduce several engineering pertinent nonlinear eigenvalue and singular-value optimization problems. The proposed approach is illustrated by means of a chemical engineering pertinent example.

2. EIGENVALUE OPTIMIZATION

In the following several eigenvalue optimization problems are described.

2.1 Maximization of the minimum eigenvalue of a symmetric matrix

The problem of maximizing the smallest eigenvalue, λ_{min}, of a symmetric matrix **A**(**y**) has been addressed in Ringetz[2]:

$$\begin{aligned} & \max_{\mathbf{y}} \lambda_{\min}(\mathbf{A}(\mathbf{y})) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \le \mathbf{0} \\ & \mathbf{y} \in \mathrm{Y} \end{aligned} \tag{P1}$$

This formulation corresponds to the problems of maximizing the lowest natural vibration frequency of a structure and of linear buckling in the field of structural engineering.

Vectors **h**(**y**) and **g**(**y**) stand for equalities and inequalities of the model respectively, and vector **y** comprises the optimization variables.

Problem (P1) can be reformulated in terms of an auxiliary variable, z:

$$\begin{aligned} & \max_{\mathbf{y},z} z \\ \text{s.t.} \quad & \lambda_i(\mathbf{A}(\mathbf{y})) \geq z \quad i = 1,...,n \\ & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in Y \end{aligned} \tag{P1'}$$

The strategy is to bound the spectrum of **A**(**y**) from below and to maximize the lower bound z. Since $\mathbf{Av} = \lambda\mathbf{Iv} \Rightarrow (\mathbf{A} - z\mathbf{I})\mathbf{v} = (\lambda - z)\mathbf{Iv}$, the condition $\lambda_i - z > 0$ ($\lambda_i > z$) implies that $\mathbf{A} - z\mathbf{I} \succ 0$ ($\succ$ represents positive definiteness). Therefore, the above problem may be rewritten as:

$$\begin{aligned} & \max_{y,z} z \\ \text{s.t.} \quad & \mathbf{A}(\mathbf{y}) - z\mathbf{I} \succ 0 \\ & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in Y \end{aligned} \tag{P1''}$$

Ringertz's approach to cope with positive definiteness makes use of the property that it is a sufficient and necessary condition for a real symmetric matrix to be positive definite that its eigenvalues be positive. Such a strategy requires a special purpose algorithm to be implemented[2].

An alternative possibility is to apply Sylvester conditions on matrix **A**. Sylvester's criterion states that the necessary and sufficient conditions that a symmetric matrix **Q**(n, n) is positive definite are that its successive principal minors $\mathbf{Q}_i$ (i=1,…,n) be positive: det[**Q**(1,1)], det[**Q**(2,2)], …, det[**Q**(n, n)]. Therefore, (P1'') can be reformulated into a new problem as:

$$\begin{aligned} & \max_{y,z} z \\ \text{s.t.} \quad & \det\{(\mathbf{A}(\mathbf{y}) - z\mathbf{I})_i\} > 0 \quad i = 1,...,n \\ & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in Y \end{aligned} \tag{P1N}$$

This is a standard NLP problem, since the determinants are themselves smooth, and can be solved with standard and efficient gradient based algorithms. The constraints of greater than zero on the determinants are handled through a small constant ξ: det(.) ≥ ξ, ξ > 0 (typically ξ=1E-5) .

2.2 Minimization of the maximum eigenvalue of a symmetric matrix

In a similar fashion, it is possible to formulate the minimization of the maximum eigenvalue of a symmetric matrix $\mathbf{A}(\mathbf{y})$:

$$\begin{aligned} & \min_{\mathbf{y}} \lambda_{max}(\mathbf{A}(\mathbf{y})) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in \mathrm{Y} \end{aligned} \tag{P2}$$

The strategy is to bound the spectrum from above with an auxiliary variable z and to minimize this upper bound. By performing analogous considerations to the maximization problem it results:

$$\begin{aligned} & \min_{\mathbf{y},z} z \\ \text{s.t.} \quad & \det\{(z.\mathbf{I} - \mathbf{A}(\mathbf{y}))_i\} > 0 \quad i = 1,...,n \\ & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in \mathrm{Y} \\ & z \leq z^u \end{aligned} \tag{P2N}$$

Again this is a regular NLP problem which can be tackled with standard NLP solvers.

2.3 Multiple objective eigenvalue optimization of a symmetric matrix

There might exist a conflict when it is desired to simultaneously maximize the minimum eigenvalue and minimize the maximum eigenvalue of a symmetric matrix. This means that while maximizing the minimum eigenvalue, its maximum eigenvalue is also maximized. Such a behavior may or may not occur depending on the involved non-linearities. It may happen for example that the maximum eigenvalue is in fact minimized when maximizing the minimum one. However, in the case when maximum and minimum eigenvalues are in conflict for minimization and maximization respectively, a multiple objective optimization problem arises:

$$\begin{aligned} & \min_{\mathbf{y}} \{\lambda_{max}(\mathbf{A}(\mathbf{y})), -\lambda_{min}(\mathbf{A}(\mathbf{y}))\} \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{y}) = \mathbf{0} \\ & \mathbf{g}(\mathbf{y}) \leq \mathbf{0} \\ & \mathbf{y} \in \mathrm{Y} \end{aligned} \tag{P3}$$

Common practice in multiple objective optimization is to reflect the tradeoff between the different objectives by constructing a non-inferior solution set. This is usually achieved by applying the "ε-constraint" method [3]. Therefore, problem (P3) can be reformulated as follows:

$$
\begin{aligned}
& \min_{\mathrm{y}} \{-\lambda_{\min}(\mathbf{A}(\mathbf{y}))\} \equiv \max_{\mathrm{y}} \lambda_{\min}(\mathbf{A}(\mathbf{y})) \\
\text{s.t.} \quad & \lambda_{\max}(\mathbf{A}(\mathbf{y})) \le \varepsilon \\
& \mathbf{h}(\mathbf{y}) = \mathbf{0} \\
& \mathbf{g}(\mathbf{y}) \le \mathbf{0} \\
& \mathbf{y} \in \mathrm{Y}
\end{aligned}
\qquad \text{(P3')}
$$

By progressive increase of the value of parameter ε, different figures of both objectives can be generated. If the constraint on the maximum eigenvalue is binding at the solution, that pair of objectives belongs to the non-inferior solution set.

Problem (P3') can be further reformulated as follows, according to (P1'):

$$
\begin{aligned}
& \max_{\mathbf{y},z} \ z \\
\text{s.t.} \quad & \lambda_i(\mathbf{A}(\mathbf{y})) \ge z \quad i = 1,\dots,n \\
& \lambda_i(\mathbf{A}(\mathbf{y})) \le \varepsilon \quad i = 1,\dots,n \\
& \mathbf{h}(\mathbf{y}) = \mathbf{0} \\
& \mathbf{g}(\mathbf{y}) \le \mathbf{0} \\
& \mathbf{y} \in \mathrm{Y}
\end{aligned}
\qquad \text{(P3'')}
$$

Applying Sylvester condition as in problems (P1N) and (P2N) it results:

$$
\begin{aligned}
& \max_{\mathbf{y},z} z \\
\text{s.t.} \quad & \det\{(\mathbf{A}(\mathbf{y}) - z.\mathbf{I})_i\} > 0 \quad i = 1,\dots,n \\
& \det\{(\varepsilon\mathbf{I} - \mathbf{A}(\mathbf{y}))_i\} > 0 \quad i = 1,\dots,n \\
& \mathbf{h}(\mathbf{y}) = \mathbf{0} \\
& \mathbf{g}(\mathbf{y}) \le \mathbf{0} \\
& \mathbf{y} \in \mathrm{Y}
\end{aligned}
\qquad \text{(P3N)}
$$

The proposed eigenvalue optimization approach provides a systematic framework to handle the whole spectrum of even medium and large-scale symmetric matrices by bounding both, minimum and maximum eigenvalues.

3. SINGULAR VALUE CONTROLLABILITY ANALYSIS

Controllability and resiliency (C&R) measures, such as singular values, condition number and Relative Gain Array are important assessment tools in dynamic and control problems [4]. They are based on linearized versions of the multiple-input/multiple-output dynamic models in the Laplace and frequency domains.

The dynamics of a process may be accurately described by a set of (generally non-linear) differential algebraic equations in the state space:

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{d})$$
$$\mathbf{y}_{\mathrm{s}} = \mathbf{g}_{\mathrm{s}}(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

where $\mathbf{u}$ is the vector of inputs, $\mathbf{y}_s$ the vector of outputs, $\mathbf{x}$ is the vector of states and $\mathbf{d}$ the vector of disturbances (all the vectors represent deviation variables from some nominal value). Performing linearization on such models gives:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{d}$$
$$\mathbf{y}_{\mathrm{s}} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{F}\mathbf{d}$$

By applying Laplace transforms to the linearized version, it is possible to obtain the transfer function representation of the system:

$$\mathbf{y}_{\mathrm{s}}(\mathrm{s}) = \mathbf{G}(\mathrm{s})\mathbf{u}(\mathrm{s}) + \mathbf{G}_{\mathbf{d}}(\mathrm{s})\mathbf{d}(\mathrm{s})$$

where

$$\mathbf{G}(\mathrm{s}) = \mathbf{C}(\mathrm{s}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{G}_{\mathbf{d}}(\mathrm{s}) = \mathbf{C}(\mathrm{s}\mathbf{I} - \mathbf{A})^{-1}\mathbf{E} + \mathbf{F}$$

In particular the minimum singular value of the steady state (zero frequency) transfer function matrix, $\mathbf{G}$:

$$\sigma_{\min}(\mathbf{G}) = \min_{\|\mathbf{u}\| \neq 0} \frac{\|\mathbf{G}\mathbf{u}\|_2}{\|\mathbf{u}\|_2}$$

indicates how close this matrix is to being singular and represents the smallest gain of the process among possible input directions. A large value of this measure implies that the process is resilient to disturbances. The singular values of matrix $\mathbf{G}$ may be calculated as the square roots of the eigenvalues of $\mathbf{H}=\mathbf{G}^{\mathrm{T}}\mathbf{G}$:

$$\sigma_i(\mathbf{G}) = \sqrt{\lambda_i(\mathbf{H})} \qquad i = 1,...,n$$

in particular

$$\sigma_{min}(\mathbf{G}) = \sqrt{\lambda_{min}(\mathbf{H})}$$

Another meaningful index, which will be introduced in order to provide a most complete description of the controllability theory, is the condition number of matrix **G**, defined as:

$$\gamma(\mathbf{G}) = \frac{\sigma_{max}(\mathbf{G})}{\sigma_{min}(\mathbf{G})}$$

which verifies the following relation:

$$\frac{\|\delta\mathbf{u}\|}{\|\mathbf{u}\|} = \gamma(\mathbf{G})\frac{\|\delta\mathbf{G}\|}{\|\mathbf{G}\|}$$

A small condition number means that model errors do not cause large manipulated variable errors.

In the following, a chemical process engineering problem will be solved in the context of formulation (P3) in order to trace the relation of the singular value structure of a matrix whose elements depend in a nonlinear fashion on the optimization variables. The condition number of the matrix is also reported in order to provide a more complete picture of the system.

4. CONTROL OF A REACTOR–SEPARATOR–RECYCLE SYSTEM

Chemical plants used to be cascades of individual units. The key for successful dynamic operation of such processes, is proper control of each unit. Modern chemical plants on the other hand are highly integrated (mass and energy recycles) in order to maximize conversion throughout the whole process and accomplish efficient energetic operation. Integrated plants present in general very complex dynamic behavior due to the positive feedback introduced by the recycles. Conventional wisdom to cope with such complex dynamics is to isolate units by means of large surge tanks, in order to reduce dynamic interaction. This practice, however, is expensive and may be environmentally unacceptable when hazardous chemicals are involved. The challenge is therefore to design for feasible operation of tightly integrated processes. Recycle systems gave rise to the concept of plantwide control, which considers processes as a whole, rather than cascades of units for control purposes[5].

In previous work[6] the cost/controllability design problem of a reactor-separator-recycle system was addressed by means of multiple objective optimization in order to illustrate their conflicting nature. In this work, a particular control configuration for the reactor-separator-recycle system, is studied in the context of problem (P3) regarding singular value controllability. The considered model is taken from Luyben[3]. The basic flow-sheet is shown in Fig. 1.

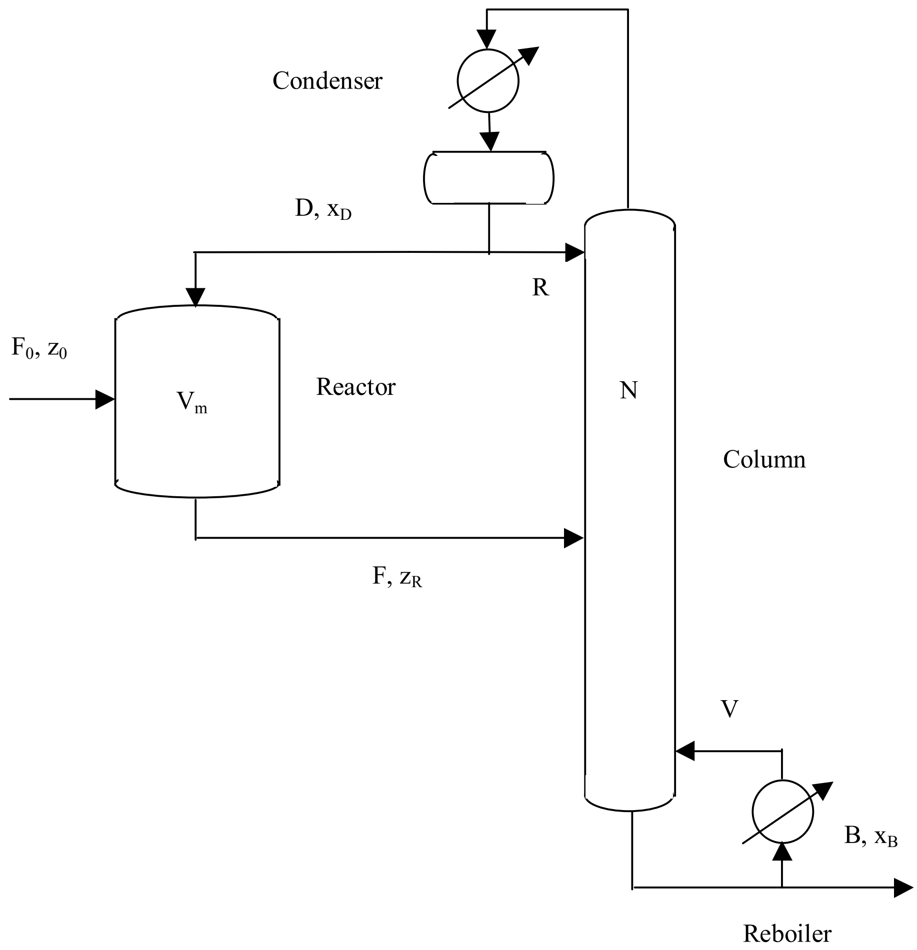

Fig. 1: Reactor-Separator-Recycle System

A first order A→B reaction takes place in the isothermal reactor. The reactor effluent is fed to a column where non-reacted A and product B are separated. Non-reacted A is recycled back to the reactor. For a given fresh feed flow-rate, F_0, and composition, z_0, and a certain product specification, x_B, the goal is to calculate the design variables, which optimize the the controllability objective. The optimization variables are: feed flow-rate, F, and composition, z_R, to the column; number of trays, N; vapor boil-up, V; reflux flow-rate, R; reflux ratio, RR; recycle flow-rate, D, and composition, x_D; and reactor hold-up, V_m, and product flow-rate, B.

Usual simplifying assumptions are considered in the column model: (i) constant relative volatility between components throughout the whole separation, (ii) equimolar over-flow, (iii) total condenser, (iv) partial reboiler and (v) saturated liquid feed. The following equations describe the mathematical model for this process. For further details see Luyben[3].

Process model

Total balance around reactor

$$F_0 + D = F \tag{m1}$$

Component balance around reactor

$$F_0 z_0 + D x_D = F z_R + V_m k z_R \tag{m2}$$

Component balance around column

$$F z_R = D x_D + B x_B \tag{m3}$$

Eduljee design equation

$$\left(\frac{N - N_m}{N + 1}\right) = 0.75\left(1 - \left(\frac{RR - RR_m}{RR + 1}\right)^{0.5668}\right) \tag{m4}$$

Minimum reflux ratio

$$RR_m = \frac{1}{\alpha - 1}\left(\frac{x_D}{z_R} - \frac{\alpha(1 - x_D)}{(1 - z_R)}\right) \tag{m5}$$

Minimum number of stages

$$N_m = \frac{\ln\left(\frac{x_D}{(1 - x_D)} \frac{(1 - x_B)}{x_B}\right)}{\ln \alpha} \tag{m6}$$

Reflux ratio

$$RR = \frac{R}{D} \tag{m7}$$

Condenser balance

$$V = R + D \tag{m8}$$

Total balance

$$F_0 = B \tag{m9}$$

Applicability of Eduljee design equation

$$RR \geq 1.0101 RR_m + 0.0101 \tag{m10}$$

Constraint on reflux concentration

$$x_D \geq 0.90 \tag{m11}$$

This model has three degrees of freedom since nine equality constraints and twelve variables are involved. The quality specification $x_D \geq 0.90$ is included in the analysis to prevent the optimal solution from having a single stripper column (R=0). α stands for relative volatility.

Economic Objective

The economic objective is the total cost to be minimized. It involves capital and utility costs:

$$\mathrm{Cost} = \frac{1}{\beta_{pay}}\left(C_{reactor} + C_{column} + C_{exchangers}\right) + \beta_{tax}\left(C_{utilities}\right)$$

where β_{pay} is the payback period and β_{tax} the tax factor. All the costs involved are in units of \$/yr.

$C_{utilities}$ mostly corresponds to the hot utility at the reboiler of the distillation column and is calculated as

$$C_{utilities} = 1207V$$

where V is supplied in kmol/hr.

The capital cost of the reactor depends on its size and is calculated here as:

$$C_{reactor} = 17639\left(D_R\right)^{1.066}\left(2D_R\right)^{0.802}$$

where

$$D_R = 0.3967\left(0.6366V_m\right)^{1/3}$$

These equations assume that the height of the reactor is twice its diameter. D_R is in meters and V_m is in kmol.

The capital cost of the column depends on the diameter, D_C, and the number of trays, N, according to the following equation:

$$C_{column} = 6802(D_c)^{1.066}(2.4N)^{0.082} + 548.8(D_C)^{1.55}N$$

The diameter depends on the vapor velocity in the column, which is related to the vapor boilup. It follows then:

$$D_C = 0.0832\sqrt{V}$$

The capital cost of heat exchangers depend upon the areas of the reboiler, A_R, and the condenser, A_C, which are at the time related to the vapor boilup:

$$C_{exchangers} = 8701 A_R^{\;0.65} + 8701 A_C^{\;0.65}$$

where

$$A_R = 0.512V \text{ and } A_C = 0.854V$$

The areas are in m^2 and V is in kmol/hr.

Steady State Gains

In order to construct matrix **G** at zero frequency, the steady state gains for the desired pairing of controlled and manipulated variables should be provided. In the present work, compositions x_D and x_B will be considered as controlled variables.

The following expressions[7] relate the aforementioned compositions to some arbitrary manipulated variable, u, and permit the calculation of the steady state gains:

$$\left(\frac{\partial x_B}{\partial u}\right)\left[\kappa_2 + \kappa_4\kappa_6 \frac{B}{V_m k} - (\kappa_1 - \kappa_4\kappa_5)\frac{B}{D}\left(1 + \frac{F}{V_m k}\right)\right] = \kappa_3 \frac{\partial RR}{\partial u} + \frac{(x_D - z_R)}{D}(\kappa_1 - \kappa_4\kappa_5)\frac{\partial D}{\partial u} \qquad (1)$$

$$\left(\frac{\partial x_D}{\partial u}\right) = \frac{(z_R - x_D)}{D}\frac{\partial D}{\partial u} - \frac{B}{D}\left(1 + \frac{F}{V_m k}\right)\frac{\partial x_B}{\partial u} \qquad (2)$$

where

$$\kappa_1 = \frac{-1}{(N+1)(\ln\alpha) x_D (1 - x_D)}$$

$$\kappa_2 = \frac{1}{(N+1)(\ln\alpha) x_B (1 - x_B)}$$

$$\kappa_3 = \frac{0.5668(RR_m + 1)}{(RR+1)(RR - RR_m)}\left[\frac{N - N_m}{N+1} - 0.75\right]$$

$$\kappa_4 = \frac{-0.5668}{(RR - RR_m)}\left[\frac{N - N_m}{N+1} - 0.75\right]$$

$$\kappa_5 = \frac{1}{(\alpha - 1)}\left(\frac{1}{z_R} + \frac{\alpha}{1 - z_R}\right)$$

$$\kappa_6 = \frac{-1}{(\alpha - 1)}\left(\frac{x_D}{{z_R}^2} + \frac{\alpha(1 - x_D)}{(1 - z_R)^2}\right)$$

With these expressions, the steady state transfer function matrix for certain control configurations can be constructed.

RV Control Configuration

In this work a singular value study of the controllability will be performed in the context of problem (P3) for the RV control configuration, where the reflux R, and the vapor boilup V, are chosen to control compositions x_D and x_B. For the sake of simplicity, perfect level control is assumed in the process (reactor, reflux drum and column base).

The desired process transfer function matrix is:

$$\mathbf{G} = \begin{bmatrix} \left(\frac{\partial x_D}{\partial R}\right)_V & \left(\frac{\partial x_D}{\partial V}\right)_R \\ \left(\frac{\partial x_B}{\partial R}\right)_V & \left(\frac{\partial x_B}{\partial V}\right)_R \end{bmatrix}$$

Its elements can be calculated from (1), (2) and the following relations:

$$\left(\frac{\partial RR}{\partial R}\right)_V = \frac{V}{D^2}, \left(\frac{\partial RR}{\partial V}\right)_R = -\frac{R}{D^2}, \left(\frac{\partial D}{\partial R}\right)_V = -1, \left(\frac{\partial D}{\partial V}\right)_R = 1$$

Singular values depend on the scaling of the variables for fair comparison[4]. Here, matrix **G** is scaled as follows:

$$\mathbf{G}^{\text{scaled}} = \begin{bmatrix} \overline{F}/(1 - \overline{x}_D) & 0 \\ 0 & \overline{F}/\overline{x}_B \end{bmatrix} \mathbf{G}^{\text{unscaled}}$$

where the overbar denotes the nominal variable value.

The parameters of the process are presented in Table 1.

Table 1: Physical Data

F_0	108.7 kmol/hr
z_0	0.9
x_B	0.00105
α	2
k	0.34086 hr^{-1}
β_{pay}	3 yr
β_{tax}	1
$\overline{F}$	125 kmol/hr
$\overline{x}_B$	0.00105

The following problem is solved to obtain the non-inferior solution set according to (P3N):

$$
\begin{aligned}
&\max_{\mathbf{y},z} z \\
\text{s.t.}\quad & \det\{(\mathbf{H}(\mathbf{y}) - z.\mathbf{I})_i\} > 0 \quad i = 1,\ldots,n \\
& \det\{(\varepsilon\mathbf{I} - \mathbf{H}(\mathbf{y}))_i\} > 0 \quad i = 1,\ldots,n \\
& \{\text{Eqs. m1 to m9}\} \\
& \{\text{Eqs. m10 and m11}\} \\
& \mathbf{y} \in Y
\end{aligned}
$$

The non-inferior solution set for $-\lambda_{min}(\mathbf{H})$ and $\lambda_{max}(\mathbf{H})$ is reported in Table 2. The corresponding singular values $\sigma_{min}(\mathbf{G})$ and $\sigma_{max}(\mathbf{G})$ are also reported in Table 2 and graphically presented in Fig. 2. The conflicting nature of both objectives can be clearly traced. The condition number $\gamma(\mathbf{G})$ is also included in Table 2 and graphed against $\sigma_{min}(\mathbf{G})$ in Fig. 3. The costs of the corresponding designs are also presented in Table 2.

The proposed technique intends to provide to the process/control designer, a picture of the criteria that should be considered regarding controllability. In particular the tradeoff between minimum-singular-value and condition-number of the process transfer function can be addressed by analyzing the reported results.

Table 2. Non inferior solution set for - $\lambda_{min}(\mathbf{H})$ and $\lambda_{max}(\mathbf{H})$

$\lambda_{min}(\mathbf{H})$	$\lambda_{max}(\mathbf{H})$	$\sigma_{min}(\mathbf{G})$	$\sigma_{max}(\mathbf{G})$	$\gamma(\mathbf{G})$	Cost ($/yr)
10149.69	200000.00	100.75	447.21	4.44	509785.95
12493.85	250000.00	111.78	500.00	4.47	510813.28
14814.17	300000.00	121.71	547.72	4.50	512208.33
17115.98	350000.00	130.83	591.61	4.52	513773.42
19402.87	400000.00	139.27	632.29	4.54	515411.73
21677.37	450000.00	147.23	670.82	4.56	517072.29
23941.36	500000.00	154.73	707.11	4.57	518727.12
26196.31	550000.00	161.85	741.62	4.58	520360.54
28443.33	600000.00	168.65	774.60	4.59	522074.22

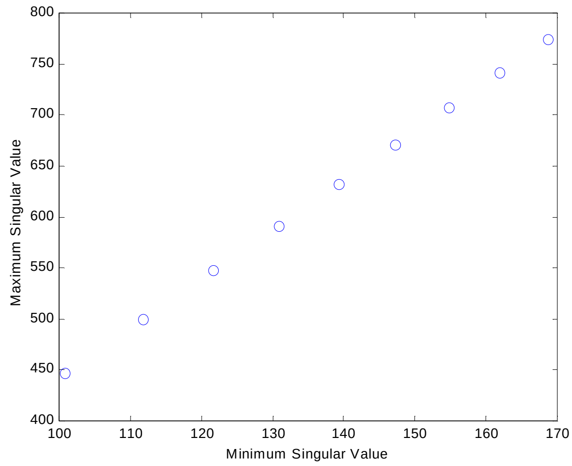

Fig. 2: Non inferior solution set for - $\sigma_{min}(\mathbf{G})$ and $\sigma_{max}(\mathbf{G})$

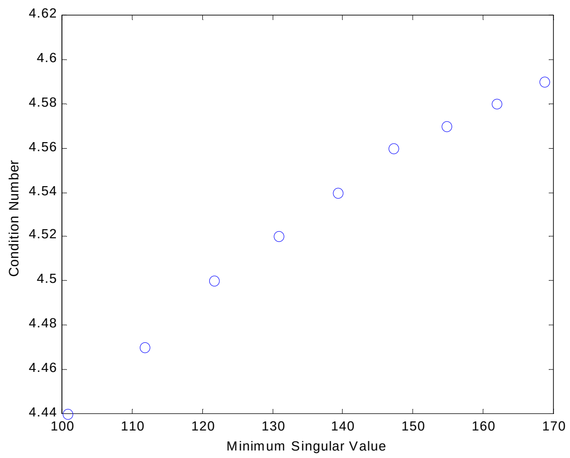

Fig. 3: $\gamma(\mathbf{G})$ vs. $\sigma_{max}(\mathbf{G})$

5. CONCLUSIONS

In this work, eigenvalue optimization theory and formulations for symmetric matrices have been introduced and applied in the context of singular value optimization. The conflicting nature of simultaneously maximizing/minimizing the minimum/maximum eigenvalues that may arise in certain cases, is addressed by means of multiple objective optimization.

Eigenvalue optimization has been applied in structural engineering problems[2] and in recent studies to chemical engineering pertinent models[6].

The proposed technique was applied here to the meaningful chemical engineering design/control problem, where the considered controllability objectives were the minimum-singular-value and the condition-number of the steady-state transfer function matrix. In particular a classic control configuration of the outstanding reactor-separator-recycle system was addressed.

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