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THE USE OF NON-CONFORMING FINITE ELEMENTS IN THREE-DIMENSIONAL VISCOUS INCOMPRESSIBLE FLOW

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Abstract. *In several papers published since the early eighties, the author demonstrated that some mixed finite element methods to solve two-dimensional viscous incompressible flow equations in primitive variables with a conforming velocity, have non-conforming three-dimensional analogues. Parallelly he established that some classical non-conforming two-dimensional methods in other formulations admit non trivial equivalent extensions to the three-dimensional case. In this work, while recalling some of the above mentionned examples, the author exhibits a case where a fundamental property of a three-dimensional non-conforming method does not hold for its analogous two-dimensional version. This property is shown to play a crucial role in connection with the Navier-Stokes equations in terms of a vector potential with a vanishing gradient on the boundary of the flow domain.*

1 NON-CONFORMING THREE-DIMENSIONAL VERSIONS OF TRIANGULAR FINITE ELEMENTS

To begin with, let us briefly recall some non-conforming tetrahedral finite elements, that can be viewed as methods equivalent to triangular ones, to solve the analogous flow equations in the corresponding case of space dimension.

First of all we consider the standard velocity-pressure formulation of the equations governing both newtonian and non-newtonian viscous incompressible flow. In this context we know that the natural and simplest possible three-dimensional version of the first mixed (conforming) finite element rigorously studied for this class of problems, is a non-conforming tetrahedron. More precisely, we mean the so-called Fortin first order triangle[4], in which the velocity is approximated by means of standard lagrangean conforming piecewise quadratic fields, and the pressure is approximated by a constant function in each triangle. As proved in [7] this element's tetrahedral version is contructed upon the same kind of pressure interpolation, while the velocity is represented through a non-conforming eight node tetrahedron, associated with incomplete quadratic fields.

Now if we switch to second order velocity-pressure methods with discontinuous pressures based on tetrahedrons, a similar situation can be observed. Indeed, consider the so-called Crouzeix and Raviart triangle [3], whose definition is recalled as follows: the velocity is approximated by means of continuous fields whose components restricted to each triangle belong to the space defined to be the direct sum of the space of quadratic functions and the cubic bubble function of the triangle. The pressure in turn is approximated by means of (discontinuous) functions, whose restriction to each triangle is linear. The velocity degrees of freedom are the values of its components at the vertices and the centers of the edges and the triangle itself.

As shown in [8], the above defined triangle admits a nonconforming three-dimensional version having equivalent properties. The velocity space for this element is the one consisting of the fields, whose restriction of each component to a given tetrahedron belongs to the space of quadratic functions in direct sum with the quartic bubble function of the tetrahedron. The pressure belongs again to the space of (discontinuous) functions whose restriction to each tetrahedron is linear. However here the velocity degrees of freedom, instead of the standard ones for classical lagrangean elements, are suitable mean values of its components along the edges and over the faces of the tetrahedron, besides their values at the barycenters of the elements. This clearly leads to non-conforming velocities.

Now, if we consider the formulation of the two-dimensional flow equations in terms of the stream function, since we are dealing with a biharmonic problem, the simplest possible finite element method that can be applied to solve them, is the so-called Morley element[6].

We recall that this is a triangular element based on quadratic functions, associated with the following degrees of freedom: the values of the function at the vertices and those of its normal derivatives at the edge mid-points. Notice that in this case we are dealing with a non-conforming triangle, and as one might expect, the three-dimensional extension of such element cannot be conforming either. As a matter of fact, using again the mean values of the function along the edges of the tetrahedrons as degrees of freedom, together with those of the normal derivatives at the barycenters of the faces, we are able to define a first order convergent finite element in the natural norm, to solve biharmonic equations in R^3 [9]. This result is entirely analogous to those that hold for the Morley element applied to the same equations in R^2. Notice that the formulation of the flow equations in three-dimension space corresponding to the one in terms of the stream function, is the vector potential formulation. This actually admits several variants, according to the set of boundary conditions satisfied by the vector potential. For instance in [10], the author studied the application of this tetrahedral finite element to the approximation of the Stokes system in terms of a vector potential, whose tangential components vanish on the boundary, assuming that the velocity also vanishes there. This element was then proved to be first order convergent in the natural norm for this problem. More recently the author attempted to apply the same method to approximate the Stokes system in terms of a vector potential, whose gradient vanishes on the boundary of the flow domain, taking again as a model, the case where the velocity vanishes on it too. In terms of function spaces, this means that we are searching for a vector potential in the Sobolev space $H_0^2(\Omega)$ (cf. [1]), where Ω denotes the flow domain with boundary $\partial\Omega$. In so doing it was found out that this extension of the Morley triangle possesses an important property that its two-dimensional companion doesn't. As pointed out before, such property turns out to be very important, provided that one is able to prove the convergence of the associated solution method for this particular type of vector potential equations. That is what we endeavour to show in the remaining sections.

2 A FORMULATION IN TERMS OF A VECTOR POTENTIAL IN $H_0^2(\Omega)$

Assume that Ω is a Lipschitz domain, and that we want to solve the following model problem:

Given $\vec{f}$ in $[L^2(\Omega)]^3$, find a velocity field $\vec{u}$ and a pressure p such that:

$$\left\{ \begin{array}{lll} \vec{u} \in [H_0^1(\Omega)]^3 & and & p \in L_0^2(\Omega), \\ -\Delta\vec{u} + \vec{grad}p & = & \vec{f} \\ div\vec{u} & = & 0, \end{array} \right. \tag{1}$$

where for a given strictly positive integer m and a sufficiently smooth Ω, $H_0^m(\Omega)$ is the subspace of $H^m(\Omega)$, consisting of functions that together with its partial derivatives up to the order $m-1$ vanish on $\partial\Omega$. We recall that $H^m(\Omega)$ is the Sobolev space of those functions belonging to $L^2(\Omega)$, whose partial derivatives up to the order m also belong to $L^2(\Omega)$, and that $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions whose integral in Ω vanish.

It is possible to prove the existence and the uniqueness of a vector potential $\vec{\psi}$ - that is, a field satisfying

$$\vec{u} = c\vec{url}\vec{\psi}, \tag{2}$$

belonging to $[H^2_0(\Omega)]^3$, and such that $\Delta^2 div\vec{\psi} = 0$ in Ω. Moreover one may establish that there exists a unique function $s \in H^1_0(\Omega)$, such that $\vec{\psi}$ and s satisfy:

$$\left\{ \begin{array}{lcl} \Delta^2\vec{\psi} - gr\vec{a}d\Delta s & = & c\vec{url}\vec{f} \\ \Delta^2 div\vec{\psi} & = & 0. \end{array} \right. \tag{3}$$

One can easily check that problem (3) may be written in the following equivalent variational form:

$$\left\{ \begin{array}{l} \text{Find } \vec{\psi} \in [H^2_0(\Omega)]^3 \text{ and } s \in H^1_0(\Omega) \text{ such that } \forall \vec{\varphi} \in \Phi \text{ and } \forall r \in H^3_0(\Omega) \\ \int_\Omega \Delta\vec{\psi} \cdot \Delta\vec{\varphi} dx - \int_\Omega gr\vec{a}ds \cdot gr\vec{a}d\ div\vec{\varphi} dx = \int_\Omega \vec{f} \cdot c\vec{url}\vec{\varphi} dx \\ \int_\Omega gr\vec{a}d\ div\vec{\psi} \cdot gr\vec{a}d\ \Delta r dx = 0 \end{array} \right. . \tag{4}$$

where $\Phi = \{\vec{\varphi}/\ \vec{\varphi} \in [H^2(\Omega) \cap H^1_0(\Omega)]^3, c\vec{url}\vec{\varphi} \in [H^1_0(\Omega)]^3\}$.

It is not so difficult either, to prove that problem (4) has a unique solution $(\vec{\psi}, s)$, namely, the solution of (3). Moreover it can be easily established that s is nothing but the divergence of $\vec{\psi}$, which a priori does not vanish. This fact is the essential reason why the uncoupled solution of the Stokes equations in terms of such vector potential should be ruled out.

In the next section we study a finite element approximation problem based on non-conforming representations of the fields involved in (4).

3 A FINITE ELEMENT SCHEME TO APPROXIMATE THE VECTOR POTENTIAL

Assume that Ω is a polyhedron, and let $\{\mathcal{T}_h\}_h$ be a quasiuniform family of partitions of Ω into tetrahedrons having a maximum edge length h, satisfying the usual compatibility conditions.

First of all we note that the simplest possible triangular finite element to approximate the stream function in two-dimensional viscous incompressible flow, is the so-called Morley triangle, whose definition was recalled in Section 1. In the same manner, the three-dimensional version of this element introduced by the author and recalled in the same Section, turns out to give rise to the simplest possible finite element representation of the vector potential. Let $\underline{\Psi}_h$ be the space of functions whose restriction to each tetrahedron of $\mathcal{T}_h$ is a polynomial of degree less than or equal to two, such that their mean values along every common edge of a patch of elements coincide, and whose first order derivatives normal to the faces of the tetrahedrons are continuous at their barycenters.

In the same way as the Morley triangle, the above mentionned tetrahedral element has the following property : The components of the gradient of every function in $\underline{\Psi}_h$ belongs to the space $\underline{S}_h$, consisting of functions whose restriction to every element of $\mathcal{T}_h$ is a polynomial of degree less than or equal to one, and that are continuous at the barycenters of the faces of the tetrahedrons. Notice that $\underline{S}_h$ is nothing but the well-known space of non-conforming piecewise linear finite elements.

Let us now turn our attention to the following question: Is there any space $\underline{R}_h$ consisting of functions whose restriction to each element of $\mathcal{T}_h$ is a polynomial of degree less than or equal to three, and whose gradient belongs to $[\underline{\Psi}_h]^3$? The answer is yes for the space $\underline{R}_h$ characterized by the continuity of the following degrees of freedom at element interfaces: The four values of the function at the vertices, the twelve mean values along the edges of the projection components of the function gradient onto the planes orthogonal to them, and the four second order normal derivatives of the function at the barycenter of the faces. In a note specified to appear shortly [5] it is proved that, given any set of twenty values of the degrees of freedom specified above, associated with a tetrahedron T, there exists a unique cubic function that corresponds to them, provided T is not degenerated (i.e. T has a strictly positive volume).

We equip $\underline{R}_h$ with the discrete H^3-norm denoted by $\| \cdot \|_{3,h}$, where for a strictly positive integer m the discrete H^m-norm denoted by $\| \cdot \|_{m,h}$ is given by:

$$\| v \|_{m,h}^2 = \sum_{T \in \mathcal{T}_h} \int_T [\sum_{i=1}^{m} |D^m v(x)|^2 dx \tag{5}$$

where $D^m v(x)$ is the m-linear form associated with the m-th order weak derivatives of v at a point x.

A first amazing thing about the above defined cubic tetrahedral finite element, is the fact that it has no two-dimensional analogue. Indeed if this happened to be the case, such cubic triangular finite element would have to be defined by a set of ten degrees of freedom. Since the derivatives of the functions in the corresponding space should belong to the finite element space associated with the Morley triangle, three of them should necessarily be the second order normal derivatives at the edge mid-points. Moreover, the two gradient components of such functions at the three vertices of every triangle should also be degrees of freedom. Indeed they must be continuous there, since the function values at those points are degrees of freedom of the Morley element. Thus we are only left one degree of freedom to define a complete cubic function in a triangle. This degree of freedom must necessarily be symmetric with respect to the three edges, and apply to the function itself, in order to allow the resulting finite element method to hold the basic approximation property for the class of methods under consideration, namely:

$$\parallel r - I_h r \parallel_{3,h} \leq Ch|r|_4 \tag{6}$$

where $I_h r$ is the interpolating function of r in $\underline{R}_h$, that is, the function whose degrees of freedom of $\underline{R}_h$ coincide with the corresponding values of r, assumed to be in $H^4(\Omega)$. As usual $|\cdot|_4$ denotes the standard seminorm of $H^4(\Omega)$ (cf. [1]), and C is a constant independent of h.

On the other hand the functions themselves belonging to such finite element space of piecewise cubics, must satisfy minimum continuity requirements at element interfaces. More specifically, according to the well-known theory of convergence applying to non-conforming finite elements (cf. [2]), in the case under study the functions are required to be continuous, whenever their restriction to each triangle is quadratic. However this can only be achieved, if there are at least three degrees of freedom per edge of a triangle related to the trace of the function on it. In the case under consideration we have two derivatives along the edges at the vertices as degrees of freedom. This indicates that such construction is unfeasible, with a sole degree of freedom available for each triangle.

Incidentally we note that the three-dimensional element holds all the required properties. In particular, the above mentionned continuity requirement is fulfilled. Indeed whenever the restriction of a function in $\underline{R}_h$ to every element of the partition is quadratic, its trace over the face of a given tetrahedron can be uniquely expressed in terms of the six degrees of freedom of the Morley triangle defined upon it (cf. [5]). Hence those traces are necessarily continuous on every face of the partition.

Let us now define the following problem to approximate (4):

$$\left\{ \begin{array}{l} \text{Find } \vec{\psi}_h \in [\Psi_h]^3 \text{ and } s_h \in S_h \text{ such that } \forall \vec{\varphi} \in \Phi_h \text{ and } \forall r \in R_h/Ker(\Delta) \\ \int_\Omega D^2\vec{\psi}_h \cdot D^2\vec{\varphi} dx - \int_\Omega \vec{grad} s_h \cdot \vec{grad}\, div\vec{\varphi} dx = \int_\Omega \vec{f} \cdot \vec{curl}\vec{\varphi} dx \\ \int_\Omega \vec{grad}\, div\vec{\psi}_h \cdot \vec{grad}\, \Delta r dx = 0, \end{array} \right. \tag{7}$$

where:

- S_h is the subspace of $\underline{S}_h$ consisting of those functions that vanish at the barycenter of every face of the partition contained in $\partial\Omega$;
- Ψ_h is the subspace of $\underline{\Psi}_h$ consisting of those functions whose degrees of freedom associated with faces of the partition contained in $\partial\Omega$ vanish;
- Φ_h is the subspace of $[\underline{\Psi}_h]^3$ consisting of those fields, whose degrees of freedom associated with every face of the partition contained in $\partial\Omega$ necessarily vanish, except the first order normal derivatives at the face barycenter, of their normal components.
- $Ker(\Delta)$ is the kernel of the discrete laplacian operator, namely, the operator whose restriction to every element of the partition is the laplacian.

It is possible to prove that problem (7) has a unique solution. However the convergence of the sequence of approximations $\{\vec{\psi}_h\}_h$ to $\vec{\psi}$ (resp. $\{s_h\}_h$ to s) in the discrete H^2-norm $\| \cdot \|_{2,h}$ (resp. in the discrete H^1-norm $\| \cdot \|_{1,h}$), can only be established if the following condition holds:

$$\left\{ \begin{array}{l} \text{There exists } C \text{ independent of } h \text{ such that} \forall t \in S_h \;\; \exists r \in R_h \\ \text{satisfying: } \Delta r = t \text{ in every } T \in \mathcal{T}_h \text{ and } \;\| r \|_{3,h} \leq C \;\; \| t \|_{1,h} \;. \end{array} \right. \tag{8}$$

To date we have not yet managed proving the validity of condition (8) under suitable assumptions on Ω, but this is actually far from obvious. Indeed its continuous counterpart is not true, since it is not possible to guarantee the existence of $r \in H^3_0(\Omega)$ such that $\Delta r = t$ for an arbitrary $t \in H^1_0(\Omega)$. However, it is not so difficult to prove that a condition analogous to (8) does hold with a constant $C(h)$ that tends to infinity as h goes to zero. Notice that if under certain assumptions (8) can be established, then in this case the vector potential $\vec{\psi} \in H^2_0(\Omega)$ will necessarily be divergence free, as the limit of the sequence $\{\vec{\psi}_h\}_h \subset [\Psi_h]^3$ consisting of fields whose divergence vanishes in every tetrahedron of $\mathcal{T}_h$. This is because we may replace Δr for r belonging to the quotient space $R_h/Ker(\Delta)$ with $t \in S_h$, in the second equation of (7), . Noticing that the discrete divergence of $\vec{\psi}_h$ belongs to S_h, taking $t = div\vec{\psi}_h$ in every tetrahedron of the partition, we readily establish that $div\vec{\psi}_h = 0$ a.e. in Ω for every h. Unfortunately this result, which would directly lead to an uncoupled solution of the vector potential linearized equations, seems to be utopic in general, if not out of reach at all.

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