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FAILURE MECHANISMS IN A TUBE UNDER VARIABLE LOADINGS

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Abstract. *The paper presents some definitions and computational procedures for the automatic recognition of mechanisms in shakedown analysis. It also contains numerical solutions for a variant of the Bree's problem consisting of a fixed-ends tube under independent variations of internal pressure and temperature, producing logarithmic instantaneous profiles across the thickness, as formulated by Gokhfeld and Cherniavsky.*[1] *In addition, automatic recognition of mechanisms is applied in this case.*

1 INTRODUCTION

Tubes under variable thermo-mechanical loadings may fail by alternating plasticity, incremental collapse or plastic collapse. We address here the identification of these mechanisms, in analytical or numerical direct solutions for the shakedown problem.

Structures subjected to variable loadings may have elastic shakedown ensured for any allowable loading program, i.e. that within the prescribed range of variable loadings. Otherwise, critical programs or cycles are included in this loading domain which produce mechanisms of impending incremental collapse or alternating plasticity.[1,3–5]

In shakedown analysis, a critical mechanism consists of a velocity field, and its compatible strain rate, together with a set of plastic strain rates associated to the extreme loads. These mechanisms can be classified as producing either pure alternating plasticity (AP), or else incremental collapse (IC). Furthermore, incremental collapse mechanisms (IC) are in turn classified as: synchronous collapse (C) (i.e. plastic collapse), simple mechanisms of incremental collapse (SMIC), or combined mechanisms of incremental collapse (CMIC). The latter mechanisms combine the accumulation of net plastic strain during each cycle and the existence of alternating plastic deformation, at one point at least.

Automatic recognition of different kinds of critical shakedown mechanisms can be implemented in numerical procedures resulting in deeper insight into the physical meaning of particular solutions. This is possible because many computational methods for direct shakedown analysis include the computation of the dual pairs of variables of the problem, together with the main result consisting of the amplifying factor for the load domain. The paper presents some definitions and computational procedures for the automatic recognition of mechanisms in shakedown analysis.

The paper contains some numerical solutions for another variant of the Bree's problem.[6] It consists of a tube under independent variations of internal pressure and temperature, producing logarithmic instantaneous profiles across the thickness, as formulated by Gokhfeld and Cherniavsky. The Mises model is adopted with the yield stress insensitive to temperature variations. The case of a tube with fixed ends is focused here, showing different failure mechanisms compared to that of the classical Bree problem. In addition, the automatic recognition of mechanisms is applied in this case.

1.1 Basic notation

The continuum model of a body is defined in an open bounded region $\mathcal{B}$ with regular boundary Γ. The space $\mathcal{V}$ is the set of all admissible velocity fields v complying with homogeneous boundary conditions prescribed on a part Γ_u of Γ. The strain rate tensor fields d are elements of the space $\mathcal{W}$, and the tangent deformation operator $\mathcal{D}$ maps $\mathcal{V}$ into $\mathcal{W}$. Let $\mathcal{W}'$ be the space of stress fields σ and $\mathcal{V}'$ the space of load systems F. The equilibrium operator $\mathcal{D}'$, dual of $\mathcal{D}$, maps $\mathcal{W}'$ into $\mathcal{V}'$. Accordingly, the kinematical and equilibrium relations are written as

$$d = \mathcal{D}v \qquad F = \mathcal{D}'\sigma \tag{1}$$

The set of all self-equilibrated (residual) stress fields is denoted by S^r.

To simplify the notation a hat is adopted to denote the local value of any field; for instance: $\hat{v} \equiv v(x)$, $\hat{d} \equiv d(x)$ and $\hat{\sigma} \equiv \sigma(x)$. Then, the internal power for any pair $\sigma \in \mathcal{W}'$ and $d \in \mathcal{W}$ is given by the duality product

$$< \sigma, d > := \int_{\mathcal{B}} \hat{\sigma} \cdot \hat{d} \; d\mathcal{B} \tag{2}$$

The stress $\hat{\sigma}$ at any point x of an elastic-ideal plastic body $\mathcal{B}$ is constrained to fulfill the plastic admissibility condition, i.e. it must belong to the set

$$P = \{\hat{\sigma} \;\mid\; f(\hat{\sigma}) \leqslant 0\} \tag{3}$$

where f is a $\hat{m}-$vector valued function describing the yield criterion. Here $\hat{m}$ is the number of yielding modes of the model (for instance: 1 for a Mises material, 6 for the Tresca criterion and 2 for a beam model including axial force). The inequality above is then understood as imposing that each component f_j of f, which is a regular convex function of $\hat{\sigma}$, is nonpositive.

Likewise, the closed convex set P of plastically admissible stress fields is

$$\sigma \in \mathsf{P} \quad \Longleftrightarrow \quad \hat{\sigma} \in P \; \forall \mathsf{x} \in \mathcal{B} \tag{4}$$

The stress–free state of the body is assumed admissible, i.e. $\hat{\sigma} = 0 \in P$.

Let us define the specific plastic dissipation function, per unit volume, as

$$\hat{\mathsf{D}}(\hat{d^p}) = \sup_{\hat{\sigma}^* \in P} \; \hat{\sigma}^* \cdot \hat{d^p} \tag{5}$$

and the indicator function $\mathcal{I}_P(\hat{\sigma})$ of P, that equals zero for any $\hat{\sigma} \in P$ and $+\infty$ otherwise. Then, the constitutive relation between plastic strain rates $\hat{d^p}$ and stresses $\hat{\sigma}$ is written,[4,7,8] for the case of associative plastic flow, as

$$\hat{\sigma} \in \partial\hat{\mathsf{D}}(\hat{d^p}) \quad \Longleftrightarrow \quad \hat{d^p} \in \mathsf{N}_P(\hat{\sigma}) \tag{6}$$

where the subdifferential $\partial\hat{\mathsf{D}}(\hat{d^p})$ is the set of all stress tensors $\hat{\sigma}$ such that

$$\hat{\mathsf{D}}(\hat{d^p}^*) - \hat{\mathsf{D}}(\hat{d^p}) \;\geqslant\; \hat{\sigma} \cdot (\hat{d^p}^* - \hat{d^p}) \quad \forall \; \hat{d^p}^* \tag{7}$$

and $\mathsf{N}_P(\hat{\sigma}) := \partial\mathcal{I}_P(\hat{\sigma})$ is the cone of normals to P at $\hat{\sigma}$, i.e. the set of all plastic strain rates $\hat{d^p}$ such that

$$(\hat{\sigma} - \hat{\sigma}^*) \cdot \hat{d^p} \geqslant 0 \quad \forall \; \hat{\sigma}^* \in P \tag{8}$$

The dissipation $\hat{\mathsf{D}}(\hat{d^p})$ can be identified as the support function of P, hence it is sublinear, i.e. convex and positively homogeneous of first degree. It also satisfies $\hat{\mathsf{D}}(0) \geqslant 0$ because $\hat{\sigma} = 0 \in P$.

The material relations (6) are equivalent to the following classical form. The plastic strain rate is related to the stress, at any point of $\mathcal{B}$, by the normality rule $\hat{d}^p = \nabla f(\hat{\sigma})\,\dot{\lambda}$. Here $\nabla f(\hat{\sigma})$ denotes the gradient of f, and $\dot{\lambda}$ is the $\hat{m}-$vector field of plastic multipliers. At any point of $\mathcal{B}$, the components of $\dot{\lambda}$ are related to each plastic mode in f by the complementarity condition: $\dot{\lambda} \geqslant 0$, $f \leqslant 0$, and $f \cdot \dot{\lambda} = 0$ (these inequalities hold componentwise).

There are global relations, in terms of the fields d^p and σ, completely analogous to the local relations (5), (6) and (7). For instance, by substituting P by P in (5) we obtain $\mathsf{D}(d^p) = \sup_{\sigma\in\mathsf{P}} < \sigma, d >= \int_{\mathcal{B}} \hat{\mathsf{D}}(\hat{d}^p)\, d\mathcal{B}$.

The total strain is the sum of elastic and plastic terms, as usual under small deformation assumptions.

2 SHAKEDOWN ANALYSIS

The data of shakedown analysis is a prescribed range of variation Δ^0 which contains any feasible history of external loads, cyclic or not. However, we prefer to represent any external action, either a mechanical or a thermal load, by the stress field which is the unique solution of the corresponding purely elastic problem. Then, the data for shakedown analysis will be given in terms of a set Δ^e of (elastic) stress fields representing the domain of variation of mechanical and thermal loads. In this paper Δ^e is assumed convex and bounded.

Moreover, all shakedown problems can be stated using the pointwise envelope Δ of the domain of elastic stresses Δ^e, which is defined in the sequel. Consider the set of all the local values of elastic stresses associated to any feasible loading, i.e.

$$\forall \mathsf{x} \in \mathcal{B} \quad \hat{\Delta} = \{\sigma^e(\mathsf{x}) \;\mid\; \forall\, \sigma^e \in \Delta^e\} \tag{9}$$

Define now the pointwise envelope of the set Δ^e

$$\Delta = \{\sigma \in \mathcal{W}' \;\mid\; \hat{\sigma} \in \hat{\Delta}\; \forall\, \mathsf{x} \in \mathcal{B}\} \tag{10}$$

As a mechanical interpretation, any (virtual) stress field σ in the set Δ may be sought as collecting local values of elastic stresses produced, at different instants, along a certain admissible load program (cyclic or not).

Any elastic field corresponding to a single feasible load also belongs to the envelope Δ (i.e. $\Delta^e \subset \Delta$). However, this envelope contains other kind of fields which, for instance, may violate the regularity conditions inherent to elastic solutions.

The fundamental theorem due to Bleich and Melan states that any load factor μ^* is safe if there exists a fixed self-equilibrated stress field σ^r such that its superposition with any stress belonging to the amplified load domain $\mu^*\Delta$ is plastically admissible. The limit load factor μ for elastic shakedown is the supremum of all safe factors; thus, the statical principle of shakedown analysis reads as follows.

Equilibrium formulation for elastic shakedown

$$\mu = \sup_{\substack{\mu^* \in R \\ \sigma^r \in \mathcal{W}'}} \left\{ \mu^* \;\middle|\; \begin{array}{l} \mu^* \Delta + \sigma^r \subset \mathsf{P} \\ \sigma^r \in S^r \end{array} \right\} \tag{11}$$

The notation used in the plastic admissibility constraint above means

$$\mu^* \Delta + \sigma^r \subset \mathsf{P} \iff \mu^* \sigma + \sigma^r \in \mathsf{P} \quad \forall\, \sigma \in \Delta \tag{12}$$

We consider in this paper the case when the loading domain is given, or approximated, by a finite number n_ℓ of basic loads, that may include thermal loadings. Furthermore, a finite element discretization of the continuum is adopted so as to produce a finite dimensional model.

Since the load domain is assumed polyhedral, then the local domain of variable loading $\hat{\Delta}$, for any point x in the body, is a convex polyhedron. The number of vertices, $n_x \leqslant n_\ell$, of this local domain depends on the point of the body considered because some of the n_ℓ elastic stress fields associated to extreme external loads (i.e. vertices of Δ^e) may produce local stress values strictly interior to the local envelope of stresses. However, we follow here the common practice of representing these local domains as the convex hull of all the n_ℓ stress values produced by the extreme loads. This is simpler and does not introduce any error, but it would be computationally advantageous in many cases to pre-process the data so as to eliminate interior (or non-extreme) stresses in the representation of local domains.

The additional step in the way to reach a finite number of admissibility constraints in Bleich-Melan's formulation is to select a discrete set $\{\mathsf{x}^j;\ j = 1, \ldots, p\}$ of critical points in the body. This is accomplished in accordance with the finite element discretization. For instance, in a mesh of n_{el} mixed or equilibrium triangles with linearly interpolated stresses the index $j = 1, \ldots, p$ enumerates all vertices of all triangles and $p = 3n_{el}$.

For the sake of simplicity we maintain the same symbols used for fields in the continuum model to denote now finite dimensional global vectors in the finite element model. For instance, $\sigma \in I\!R^q$ denotes in this subsection the global column vector of interpolation parameters for stress fields in an statical or mixed formulation. Accordingly, σ^r is the finite dimensional vector of global residual stresses. If we use n_{el} mixed triangles where $\hat{q}$ stress components are linearly interpolated in terms of nodal parameters, then each column vector σ or σ^r has $q = 3n_{el}\hat{q}$ components.

In the sequel, the local domain of variable stresses $\hat{\Delta}(\mathsf{x}^j)$, which is a convex polyhedron for any $j = 1, \ldots, p$, is described as the convex hull of its n_ℓ vertices, now written as global column vectors. Additionally, a single index $k = 1, \ldots, m$ is used to enumerate all local vertices in all the finite elements of a mesh (i.e. $m = pn_\ell$). Then, the uncoupled envelope of the elastic stresses is written as

$$\Delta = \mathrm{co}\,\{\sigma^k;\ k = 1, \ldots, m\} \tag{13}$$

where σ^k denotes the global stress vector representing a vertex of some $\hat{\Delta}(\mathsf{x}^j)$ and thus also a vertex of Δ. This global vector is composed of the elastic stress produced by a single load case in one of the selected points of some element and completed with zeros for all other components (this is only to simplify the theoretical presentation).

Accordingly, the plastic admissibility condition $\mu\Delta + \sigma^r \subset \mathsf{P}$ is equivalent to the following constraints

$$\mu\sigma^k + \sigma^r \in \mathsf{P}^k \quad k = 1, \ldots, m \tag{14}$$

where P^k represents the elastic range of a particular point in the continuum, written in terms of the global vector of stress parameters, i.e. previously selecting the pertinent components.

Finally, the discrete forms of compatibility and equilibrium created by the finite element discretization read

$$d = Bv \qquad\qquad B^T\sigma^r = 0 \tag{15}$$

Consequently, the Bleich–Melan's formulation for this case can be simplified as follows.

Discrete equilibrium formulation for elastic shakedown

$$\mu = \sup_{\substack{\mu^* \in R \\ \sigma^r \in R^q}} \left\{ \mu^* \;\middle|\; \begin{array}{ll} \mu^*\sigma^k + \sigma^r \in \mathsf{P}^k & k = 1, \ldots, m \\ B^T\sigma^r = 0 & \end{array} \right\} \tag{16}$$

This optimization problem may be recast in several ways.[2,9] In particular, the stationary conditions below are specially useful.

Discrete optimality conditions for elastic shakedown:

$$B^T\sigma^r = 0 \tag{17}$$

$$\sum \dot{\lambda}^k \, \nabla f(\mu\sigma^k + \sigma^r) = Bv \tag{18}$$

$$\sum \dot{\lambda}^k \, \sigma^k \cdot \nabla f(\mu\sigma^k + \sigma^r) = 1 \tag{19}$$

$$\dot{\lambda}^k f(\mu\sigma^k + \sigma^r) = 0 \qquad k = 1, \ldots, m \tag{20}$$

$$f(\mu\sigma^k + \sigma^r) \leqslant 0 \qquad k = 1, \ldots, m \tag{21}$$

$$\dot{\lambda}^k \geqslant 0 \qquad k = 1, \ldots, m \tag{22}$$

We use in the applications a general iterative algorithm for the above discrete problem presented in.[10] The algorithm is based on the set of discrete optimality conditions, and it is performed in two steps per iteration. In the first step a Newton iteration for the subset of equalities is performed. The second step consists of relaxation and uniform scaling of stresses in order to maintain plastic feasibility, represented by the inequalities in the optimality conditions.

3 FAILURE MECHANISMS IN SHAKEDOWN

Let us denote the solution of the elastic shakedown problem by μ, v, σ^r and $\{d^{j\ell};\ j = 1, ..., p\,;\ \ell = 1, ..., n_F\}$ where p is the total number of selected control points in the mesh and n_ℓ the number of extreme loads.

The type of failure mechanism is then identified as follows.

AP Alternate plasticity: The compatible strain rate vanishes, i.e.

$$Bv = 0 \tag{23}$$

IC Incremental collapse: The compatible strain rate is not zero, i.e.

$$Bv \neq 0 \tag{24}$$

Solutions complying with the above condition can be further classified as follows.

C Plastic collapse: There is no more than one possible nonzero plastic strain rate $d^{j\bar{\ell}}$ among the set of all $d^{j\ell}$ at each control point x^j, and all these nonzero strain rates correspond to the same extreme load $\bar{\ell}$, i.e. $\forall\ j = 1, ..., p$

$$d^{j\ell} = 0 \quad \forall \quad \ell = 1, ..., n_\ell \quad \text{and such that} \quad \ell \neq \bar{\ell} \tag{25}$$

Due to the relation $Bv = \sum d^{j\ell}$, the above condition implies that the compatible strain rate coincides locally, for each control point x^j, with the plastic strain rate $d^{j\bar{\ell}}$. Also, by using the local plastic flow relation $\mu d^{j\ell} + \sigma^r \in \partial\chi^j(d^{j\ell})$, it follows that

$$\sigma^c := \mu\sigma^{\bar{\ell}} + \sigma^r \in \partial\chi(Bv) \tag{26}$$

The stress $\sigma^{\bar{\ell}}$ above, collects the local values given by $\sigma^{j\bar{\ell}}$; thus, it is the purely elastic stress associated to the load $\bar{\ell}$. Consequently, the resulting critical stress above, σ^c, produces instantaneous collapse.

SMIC Simple mechanism of incremental collapse: There is no more than one possible nonzero plastic strain rate $d^{j\hat{\ell}}$ among the set of all $d^{j\ell}$ at each control point x^j, i.e. $\forall\ j = 1, ..., p$

$$d^{j\ell} = 0 \quad \forall \quad \ell = 1, ..., n_\ell \quad \text{and such that} \quad \ell \neq \hat{\ell} \equiv \hat{\ell}(j) \tag{27}$$

Due to the relation $Bv = \sum d^{j\ell}$, the above condition implies that the compatible strain rate coincides locally, for each control point x^j, with the corresponding plastic strain rate $d^{j\hat{\ell}}$. Then, by using also $\mu\sigma^{j\ell} + \sigma^r \in \partial\chi^j(d^{j\ell})$, it follows that there exists a global stress $\sigma \in \Delta$ such that

$$\mu\sigma + \sigma^r \in \partial\chi(Bv) \tag{28}$$

The stress σ above, collects the local values given by $\sigma^{j\hat{\ell}}$. Thus, it represents a critical loading cycle that activates no more than one extreme load at each point of the body. It produces an (impending) monotonous increase in plastic strains, i.e. a non-synchronous collapse.

Plastic collapse (or instantaneous collapse) is the particular case of SMIC where at all points in the body the same load is active, i.e. $\hat{\ell}(j) = \overline{\ell}$ (constant).

CMIC Combined mechanism of incremental collapse: At one point of the body, at least, there are more than one nonzero plastic strain rates among the set of all $d^{j\ell}$ corresponding to that point, i.e. there exist some $\hat{j}, \ell_1$ and ℓ_2 such that

$$\|d^{\hat{j}\ell_1}\| > 0 \quad \text{and} \quad \|d^{\hat{j}\ell_2}\| > 0 \tag{29}$$

4 A RESTRAINED TUBE UNDER VARIABLE TEMPERATURE AND PRESSURE

We present in this section numerical solutions for a fixed-end thick tube submitted to independent variations of internal pressure and temperature. Due to the axial restraint the simple mechanisms of incremental collapse, that solve exactly the closed-end tube considered in the classical Bree problem, are no longer critical, as detected in the finite element solution. The instantaneous temperature pattern is logarithmic across the wall thickness, and vanishes cyclically. The material behaves following von Mises model. This case, considered, with closed ends, by Gokhfeld and Cherniavsky[1] (at pages 167-175), is a variant of the fundamental Bree problem[6,11,12]

Consider a long tube with fixed ends. The internal and external radii are R_{int} and R_{ext}, respectively. The radial coordinate R is substituted by the dimensionless radius r given below, together with the relevant geometric parameter ℓ.

$$r := \frac{R}{R_{ext}} \qquad\qquad \ell := \frac{R_{ext}}{R_{int}} \tag{30}$$

The internal pressure p_{int} varies between 0 and $\overline{p_{int}}$. Accordingly, the dimensionless mechanical parameter is defined as

$$p := \frac{p_{int}}{(\ell^2 - 1)\,\sigma_Y} \tag{31}$$

where σ_Y denotes the yield stress. Then, p varies between 0 and $\overline{p} := \overline{p_{int}}/(\ell^2 - 1)\,\sigma_Y$.

Plastic collapse of the tube is produced at the following internal pressure

$$p_c = \frac{2}{\sqrt{3}}\,\sigma_Y\ \ln \ell \tag{32}$$

This suggests the use of an additional dimensionless parameter, defined as

$$\hat{p} := \frac{p_{int}}{p_c} = \sqrt{3}\,\beta\, p \tag{33}$$

varying between 0 and $\underline{\hat{p}} := \sqrt{3}\,\beta\,\overline{p}$. We used above, for convenience, the expression

$$\beta := \frac{\ell^2 - 1}{2\,\ln \ell} \tag{34}$$

Notice that $\ell < \beta < \ell^2$ because $\ell > 1$. Further, the approximations representing thin tubes are obtained for $\ell \to 1^+$, that implies $\beta \to 1^+$.

Independently from pressure, the difference between internal and external wall temperatures, $\Theta_{int} - \Theta_{ext}$, varies between 0 and $\overline{\Theta}$. The temperature at a distance r of the axis is assumed to follow, at any instant, the steady state pattern:

$$\Theta = \Theta_{ext} - (\Theta_{int} - \Theta_{ext})\,\frac{\ln r}{\ln \ell} \tag{35}$$

Then, a suitable dimensionless thermal parameter is

$$q := \frac{\mathsf{E}\,c_\Theta\,(\Theta_{int} - \Theta_{ext})}{2\,\sigma_Y\,(1-\nu)\,(\ell^2 - 1)} \tag{36}$$

where E denotes the Young's modulus, ν is the Poisson's coefficient, and c_Θ is the thermal expansion coefficient. Consequently, the prescribed limits for temperature loading are 0 and $\overline{q} := \mathsf{E}\,c_\Theta\,\overline{\Theta}/2\,\sigma_Y\,(1-\nu)\,(\ell^2-1)$, in dimensionless form.

In order to produce Bree–type diagrams in the usual standards, we define the additional dimensionless thermal parameter

$$\hat{q} := \frac{\mathsf{E}\,c_\Theta\,(\Theta_{int} - \Theta_{ext})}{2\,\sigma_Y\,(1-\nu)} = (\ell^2 - 1)q \tag{37}$$

with bounds 0 and $\underline{\hat{q}} := (\ell^2 - 1)\overline{q} = \mathsf{E}\,c_\Theta\,\overline{\Theta}/2\,\sigma_Y\,(1-\nu)$.

External loading for the tube is given, in shakedown analysis, by the elastic stress solutions: T^p, under pure pressure, and T^q, under pure thermal loading. These stress fields are given below, in dimensionless form, by using the reduced stress tensors $\widetilde{T} := (1/\sigma_Y)T$, $\widetilde{T}^p := (1/p\sigma_Y)T^p$, and $\widetilde{T}^q := (1/q\sigma_Y)T^q$. Accordingly, variable loading produce the following elastic stress

$$\widetilde{T} = p\widetilde{T}^p + q\widetilde{T}^q \tag{38}$$

where the basic elastic fields are:

The four basic loadings $T^k(r)$ are determined from the following stress fields:

(i) Elastic stresses due to pressure loading

$$\widetilde{T}^p_r = 1 - r^{-2} \qquad \widetilde{T}^p_\theta = 1 + r^{-2} \qquad \widetilde{T}^p_z = 2\nu \tag{39}$$

(ii) Elastic stresses due to temperature loading

$$\widetilde{T}^q_r = r^{-2} - 1 + 2\beta \ln r \tag{40}$$

$$\widetilde{T}^q_\theta = -r^{-2} - 1 + 2\beta\,(1 + \ln r) \tag{41}$$

$$\widetilde{T}^q_z = 2\,[\nu\,(\beta - 1) + 2\beta \ln r] \tag{42}$$

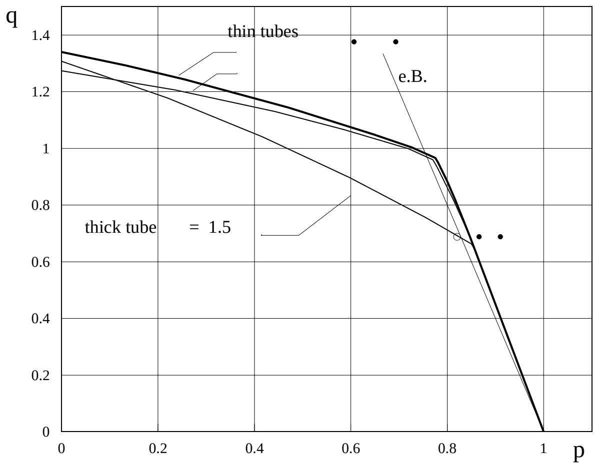

Figure 1: Shakedown boundaries for fixed-end thick tubes under independent variations of pressure and (logarithmic) temperature. Bree–type diagram: thermal load $\hat{q}$, (37), versus pressure $\hat{p}$, (33). Line e.B.: extended Bree solution. Dots: specific cycles producing incremental collapse, obtained by Hyde *et al.*[13] Circles: extrapolated ratcheting boundary for thin tubes, predicted by Hyde *et al.*

The local domain of variable loading, $\Delta(r)$, is a parallelogram with four vertices $\{\widetilde{T}^k(r);\ k = 1, \ldots, 4\}$ given by (38) with $(p, q) = \{(0, 0), (0, \overline{q}), (\overline{p}, \overline{q}), (\overline{p}, 0)\}$.

The finite element procedure for shakedown analysis is applied to thin tubes ($\ell \leqslant 1.1$) and a thick tube with $\ell = 1.5$. The results are shown in Figure 1.

Two branches are clearly defined in the diagram of each tube. The portion of the boundary with high pressures (close to the collapse pressure) correspond to combined mechanisms of incremental collapse (CMIC). The remaining part, with high temperatures, present alternating plasticity (AP) as the critical failure mechanism. All these failure mechanisms are correctly detected by the numerical procedures presented above.

5 CONCLUSIONS

A finite element procedure for shakedown of structures is equipped with an automatic tool to identify failure mechanisms. The solving algorithm and the recognition procedure are applied to a restrained tube, under variable thermo-mechanical loadings.

The finite element solution for the restrained tube, which is another variant of the Bree problem, is in good agreement with the specific incremental collapse cyclic loads reported by Hyde *et al.*[13]

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