Mecánica Computacional Vol. XXI, pp. 1402–1413
S. R. Idelsohn, V. E. Sonzogni and A. Cardona (Eds.)
Santa Fe–Paraná, Argentina, October 2002

GOOD QUALITY POINT SETS AND MOVING LEAST SQUARE APPROXIMATIONS

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Abstract. *The goal of this paper is to study the relation of the condition numbers of the star of nodes in normal equations for error estimates of Moving Least Square approximations in Sobolev spaces. The condition numbers are closely related to the good quality of the set of nodes and the approximating power of the method.*
Key Words: meshless methods, moving least square interpolation, error estimates.

1 INTRODUCTION

The moving least square (MLS) as approximation method has been introduced by Shepard[1] in the lowest order case and generalized to higher degree by Lancaster and Salkauskas[2]. The use of MLS in solving PDEs was pioneered by the works of B. Nayroles, T. Belytschko and others[3,4,5,6].

For this kind of applications it is fundamental to analyze the order of approximation, not only for the function itself, but also for its derivatives. C. Zuppa[7] introduced *condition numbers* of the *stars* of *nodes* in the normal equation which are closely related to the approximation power of the method in the space of differentiable functions. We present here some theoretical analysis of the *condition numbers* and error estimates in the framework of Sobolev spaces[8].

In this paper is discussed the close relation of the *condition numbers* with the good quality point set generation problem for meshless methods. For numerical simulation problems, meshless methods have emerged as an alternative to mesh based methods for several reasons, among them, because mesh generation is a difficult problem, and these methods have generated promising results in the accuracy of simulations. Ealier implementation of meshless methods employed arbitrary set of *nodes,* but now it is recognized that an inteligent placement of the *nodes* it is necessary to avoid the under-sampling of important physical phenomena. In X. Y. Li *et al.* [9,10,11,12] a set of criteria is formalized to define good selections of *nodes* in the domain in order to increases accuracy of solution. The practical importance of the *condition number* for good quality point sets are discussed.

2 PRELIMINARIES

In the n-dimensional space $\mathbb{R}^n$ let $||\,.\,||$ denote the Euclidean norm and $B_r(\mathbf{y})$ denote the open ball $\{\,\mathbf{x}\in\mathbb{R}^n|\ ||\mathbf{x}-\mathbf{y}||<r\,\}$ with center $\mathbf{y}$ and radius r. We use standard multi-index notation. In particular, given any multi-index $\nu=(\nu_1,...,\nu_n)\in\mathbb{N}^n$, $|\nu|$ denotes the sum $\nu_1+...+\nu_n$, and, if f is a sufficiently smooth function, $D^\nu f$ denote the partial derivative $\frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}...\partial x_n^{\nu_n}}f$.

Let Ω be an open bounded domain in $\mathbb{R}^n$ and Q_N denote an arbitrarily chosen set of N points $x_\alpha\in\overline{\Omega}$ referred to as *nodes*:

$$Q_N=\{\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_N\}\,,\quad \mathbf{x}_\alpha\in\overline{\Omega}$$

Let $\mathcal{I}_N:=\{\omega_\alpha\}_{\alpha=1}^N$ denote a finite open covering of $\overline{\Omega}$ consisting of N *clouds* ω_α such that $\mathbf{x}_\alpha\in\omega_\alpha$ and ω_α is 'centered ' around $\mathbf{x}_\alpha$ in some way, and

$$\overline{\Omega}\subset\bigcup_{\alpha=1}^{N}\omega_\alpha, \tag{1}$$

A class of functions $\mathcal{S}_N:=\{\mathcal{W}_\alpha\}_{\alpha=1}^N$ is called a partition of unity subordinated to the open covering $\mathcal{I}_N$ if it possesses the following properties:

- $\mathcal{W}_\alpha \in C_0^s(\mathbb{R}^n), \quad s \geq 0$ or $s = +\infty$
- $supp(\mathcal{W}_\alpha) \subseteq \overline{\omega}_\alpha$
- $\mathcal{W}_\alpha(\mathbf{x}) > 0, \quad x \in \omega_\alpha$
- $\sum_{\alpha=1}^{N} \mathcal{W}_\alpha(\mathbf{x}) = 1$, for every $\mathbf{x} \in \overline{\Omega}$.

There is no unique way to build a partition of unity as defined above. A widely used approach in practice is the following:

For each $\alpha = 1, ..., N$, ω_α is an open ball $B_{d_\alpha}(\mathbf{x}_\alpha)$ such that 1 is verified. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^s-function such that $\varphi(\mathbf{x}) > 0$ if $\mathbf{x} \in B_1(\mathbf{0})$ and $supp(\varphi) = \overline{B_1(\mathbf{0})}$. For $\alpha = 1, ..., N$, let us define functions ψ_α by formula

$$\psi_\alpha(\mathbf{x}) = \varphi\left(\frac{\mathbf{x} - \mathbf{x}_\alpha}{d_\alpha}\right)$$

and $\mathcal{W}_\alpha$ by

$$\mathcal{W}_\alpha(\mathbf{x}) = \frac{\psi_\alpha(\mathbf{x})}{\sum \psi_\beta(\mathbf{x})} \tag{2}$$

We shall mainly be concerned in this paper with this kind of partition of unity and we assume also that $s \geq 2$. From now on, let $A > 0$ be a constant such that

$$||\nabla\varphi||_{L^\infty(\mathbb{R}^n)} \leq A \tag{3}$$

Definition 2.0.1 *For any $\alpha = 1, ..., N$, we set*

$$n(\alpha) := \{\beta | \, \omega_\alpha \cap \omega_\beta \neq \emptyset\}$$

The following conditions on the partition of unity will be assumed from now on.

(H1) The diameter of the *clouds* are *locally comparable*, i.e., there is $C_1, C_2 > 0$ such that $\forall\, \alpha$, $\alpha = 1, ..., N$,

$$C_1 d_\beta \leq d_\alpha \leq C_2 d_\beta \qquad \forall \beta \in n(\alpha) \tag{4}$$

(H2) The overlap of neighboring clouds are controlled by $M \in \mathbb{N}$. That is

$$\forall \alpha,\ \alpha = 1, ..., N, \qquad \#\{n(\alpha)\} \leq M \tag{5}$$

(H3) There is $D > 0$ such that

$$D \leq \sum \psi_\beta(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \overline{\Omega} \tag{6}$$

The $i-$partial derivative of $\mathcal{W}_\alpha$ is

$$\frac{\partial \mathcal{W}_\alpha}{\partial x_i} = \frac{\frac{\partial \psi_\alpha}{\partial x_i}\left(\sum \psi_\beta\right) - \psi_\alpha\left(\sum \frac{\partial \psi_\beta}{\partial x_i}\right)}{\left(\sum \psi_\beta\right)^2}$$

Using (3), (4), (5) and (6) it is easily shown that

(H4) There is $C_G > 0$ such that

$$||\nabla \mathcal{W}_\alpha||_{L^\infty(\overline{\Omega})} \leq \frac{C_G}{d_\alpha} \qquad \forall \alpha,\ \alpha = 1, ..., N \tag{7}$$

Condition (H1) implies that there exist a parameter $d > 0$ such that

$$\widetilde{C}_1 d \leq d_\alpha \leq \widetilde{C}_2 d \qquad \forall \alpha \tag{8}$$

and this appears somewhat restrictive. It implies in particular that grid size chage smoothly. The more general case of arbitrary support size will be studied in a forthcoming paper.

3 THE MOVING LEAST SQUARE METHOD

Given data values $\mathbf{f} = (f_\alpha)_{\alpha=1}^N$ at *nodes* x_α, the MLS method produces a function $\widehat{f} \in C^s(\mathbb{R}^n)$ that interpolates data $\mathbf{f}$ in a weighted square sense. Let $\mathcal{P}_q$ the space of polynomial of degree q, $q \ll N$ and $q \leq s$, and let $\mathcal{B}_q = \{p_0, p_1, ..., p_m\}$ be any basis of $\mathcal{P}_q$. For each $\mathbf{z} \in \overline{\Omega}$ (fixed) we consider

$$P^*(\mathbf{z}, \mathbf{x}) = \sum_{0 \leq j \leq m} a_j(\mathbf{z})\ p_j(\mathbf{x})$$

where $\mathbf{a} = \{a_j(\mathbf{z})\}_{0 \leq j \leq m}$ are chosen such that

$$J_{\mathcal{B}_q, z}(\mathbf{a}) = \frac{1}{2} \sum_{\alpha=1}^N \mathcal{W}_\alpha(\mathbf{z}) \left(\sum_{0 \leq j \leq m} a_j\, p_j(\mathbf{x}_\alpha) - f_\alpha \right)^2 \tag{9}$$

is minimized. Then, we define the approximation $\widehat{f}$ in $\mathbf{z}$ by

$$\widehat{f}(\mathbf{z}) = P^*(\mathbf{z}, \mathbf{z})$$

Definition 3.0.2 *Given* $\mathbf{z} \in \overline{\Omega}$, *the set* $\mathcal{ST}(\mathbf{z}) = \{\alpha | \mathcal{W}_\alpha(\mathbf{z}) \neq 0\}$ *will be called the star at* $\mathbf{z}$.

It is clear that the sum in (9) is extended only over the set $\mathcal{ST}(\mathbf{z})$. The set of *nodes* $\mathcal{SN}(\mathbf{z})$ in the *star* $\mathcal{ST}(\mathbf{z})$ is $\{\mathbf{x}_\alpha | \alpha \in \mathcal{ST}(\mathbf{z})\}$. If $\mathcal{SN}(\mathbf{z}) = \{\mathbf{x}_{\alpha_1}, ..., \mathbf{x}_{\alpha_K}\}$, for the sake of simplicity we shall denote this set as $\{\mathbf{x}_1, ..., \mathbf{x}_K\}$. Observe that the polynomial $P^*(\mathbf{z}, \mathbf{x})$ can be obtained by solving the normal equations for the minimization problem. In fact, if we denote

$$F(\mathcal{B}_q) = \begin{pmatrix} p_0(\mathbf{x}_1) & p_0(\mathbf{x}_2) & \cdots & p_0(\mathbf{x}_K) \\ p_1(\mathbf{x}_1) & p_1(\mathbf{x}_2) & \cdots & p_1(\mathbf{x}_K) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_1) & p_m(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_K) \end{pmatrix},$$

$$W(\mathbf{z}) = \begin{pmatrix} \mathcal{W}_1(\mathbf{z}) & 0 & \cdots & 0 \\ 0 & \mathcal{W}_2(\mathbf{z}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_K(\mathbf{z}) \end{pmatrix}$$

then, $\mathbf{a} = (a_0(\mathbf{z}), ..., a_m(\mathbf{z}))$ is the solution of the following system:

$$F(\mathcal{B}_q) W(\mathbf{z}) F^T(\mathcal{B}_q)\, \mathbf{a} = F(\mathcal{B}_q) W(\mathbf{z})\, \mathbf{f} \tag{10}$$

In order to have the moving least square approximation well defined we need the minimization problem to have a unique solution at every $\mathbf{z} \in \overline{\Omega}$ and this is equivalent to the non-singularity of matrix $F(\mathcal{B}_q)W(\mathbf{z})F^T(\mathcal{B}_q)$. Error estimates are obtained under the following assumption about the system of *nodes* and weight functions $\{\mathcal{Q}_N, \mathcal{S}_N\}$:
Property $\mathbf{R}_q$: for any $\mathbf{z} \in \overline{\Omega}$, the normal matrix $F(\mathcal{B}_q)W(\mathbf{z})F^T(\mathcal{B}_q)$ is non singular.

Definition 3.0.3 *If $\#\mathcal{P}_q = N_q$, a set of nodes $\{\mathbf{x}_j \in \mathbb{R}^n : j = 1, ..K\}$ is called $\mathcal{P}_q-$unisolvent if the Vandermondian*

$$F(\mathcal{B}_q) = \begin{pmatrix} p_0(\mathbf{x}_1) & p_0(\mathbf{x}_2) & \cdots & p_0(\mathbf{x}_K) \\ p_1(\mathbf{x}_1) & p_1(\mathbf{x}_2) & \cdots & p_1(\mathbf{x}_K) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_1) & p_m(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_K) \end{pmatrix}$$

has range equal to N_q.

It is clear that this property does not depend on the basis $\mathcal{B}_q$.

The next theorem[8], gives us a necessary and sufficient condition for the satisfaction of property $\mathbf{R}_q$ in a stable way. It should be remarked that a related issue was considered by W. Han and X. Meng[13] in the context of approximations based also on partition of unity.

Theorem 3.0.4 *A necessary and sufficient condition for the satisfaction of* ***Property*** $\boldsymbol{R}_q$ *is that, for any* $\mathbf{z} \in \overline{\Omega}$, *the set*

$$\{\, \mathbf{x}_{\alpha_k} \,|\, \alpha_k \in \mathcal{ST}(\mathbf{z})\}$$

is $\mathcal{P}_q-$*unisolvent.*

Let $\mathcal{F} := \mathbb{R}^N$ be the set of possible values $\mathbf{f} = (f_\alpha)_{\alpha=1}^N$ of functions at the *nodes* x_α. Under the assumption above, the MLS method provides an operator $\mathcal{A} : \mathcal{F} \to C^s(\overline{\Omega})$ defined by

$$\mathcal{A}(\mathbf{f})(\mathbf{z}) = \widehat{f}(\mathbf{z}), \qquad \mathbf{f} \in \mathcal{F}, \quad \mathbf{z} \in \overline{\Omega}$$

This is not an interpolation operator in the sense that, in general, $\mathcal{A}(\mathbf{f})(\mathbf{x}_\alpha) \neq f_\alpha$.

Given a function $f \in C^p(\overline{\Omega})$, the associated vector in $\mathcal{F}$ is $\mathbf{f} = (f(\mathbf{x}_\alpha))_{\alpha=1}^N$ and we shall write $\mathcal{A}(f)$ for $\mathcal{A}(\mathbf{f})$. The operator $\mathcal{A}$ is linear and $q-$*reproductive*, that is, $\mathcal{A}(P) = P$ when P is a polynomial of degree q.

Given $\alpha \in \{1, ..., N\}$, let $\mathbf{e}_\alpha = (0, ..., 0, 1, 0, ...0)$ the vector in $\mathbb{R}^N$ which has a unique coordinate distint of zero and equal to one at the $\alpha^{th}-$place, and $\phi_\alpha = \mathcal{A}(\mathbf{e}_\alpha)$. The set of functions $\{\phi_\alpha\}_{\alpha=1,...N}$ are the canonical shape functions associated to the approximation operator $\mathcal{A}$ in the sense that, for every $\mathbf{f} \in \mathcal{F}$ we have

$$\mathcal{A}(\mathbf{f}) = \sum_{\alpha=1}^{N} f_\alpha \cdot \phi_\alpha \tag{11}$$

A fundamental ingredient in establishing error estimates in moving least square approximations is to obtain bounds for functions $\{\phi_\alpha\}$ and its derivatives.

Property $\mathbf{R}_q$ does not depend on the basis of $\mathcal{P}_q$ and this property will play a fundamental role in our work. In fact, if $\mathcal{A}_q = \{q_0, q_1, ..., q_m\}$ is another basis of $\mathcal{P}_q$ such that $\mathcal{B}_q = G\mathcal{A}_q$, G being a non-singular matrix, then

$$J_{\mathcal{A}_q,z} = J_{\mathcal{B}_q,z} \circ G, \qquad \mathbf{z} \in \overline{\Omega}$$

and $J_{\mathcal{A}_q,z}$ has a unique minimum if and only if $J_{\mathcal{B}_q,z}$ does.

Therefore, in analyzing the normal equation in a neighborhood of a given point $\mathbf{z} \in \overline{\Omega}$, we can choose a convenient basis. In our work, this basis will be the Taylor monomial centered at $\mathbf{z}$:

$$\mathcal{T}_{\mathbf{z}}^q = \{(\mathbf{x} - \mathbf{z})^\eta\}_{0 \leq |\eta| \leq q}$$

Assumption. In what follows, we shall deal only with case $q = 1$. This case is the most used in practice because of the notorious snaking polynomial problem. The reader should bear in mind however that results could be generalized to degrees higher than one.

3.1 The derivatives of $\mathcal{A}(\mathbf{f})$

Given $\mathbf{f} \in \mathcal{F}$, for each $\mathbf{c} \in \overline{\Omega}$ (fixed) we want to make explicit, following Zuppa[2], the formulae of the derivatives of $\mathcal{A}(\mathbf{f})$ at $\mathbf{c}$ that will be useful in future calculations.

In all what follow we shall use the basis $\mathcal{T}_{\mathbf{c}}^{q} = \{(\mathbf{x}-\mathbf{c})^{\eta}\}_{0\leq|\eta|\leq q}$ of $\mathcal{P}_q$, $q = 1, 2$, and, in order to simplify notation, we will drop any reference to this basis in the normal equation. Therefore, we have

$$\mathcal{A}(\mathbf{f})(\mathbf{x}) = \sum_{0\leq|\eta|\leq q} a_{\eta}(\mathbf{x})(\mathbf{x}-\mathbf{c})^{\eta} \tag{12}$$

where $\mathbf{a} = (a_{\eta}(\mathbf{x}))_{0\leq|\eta|\leq q}$ is the solution of :

$$FW(\mathbf{x})F^{T}\,\mathbf{a} = FW(\mathbf{x})\,\mathbf{f} \tag{13}$$

In order to calculate the values $D^{\eta}\mathcal{A}(\mathbf{f})(\mathbf{c})$, $0 \leq |\eta| \leq 1$, it is useful to use the following notation:

- $\mathbf{0} \in \mathbb{R}^N$ is the multi-index $(0, 0, ..., 0)$.
- For $i = 1, ..., n$, $\mathbf{e}_i$ is the multi-index with $|\mathbf{e}_i| = 1$, $\mathbf{e}_i = (0, ..., 1, ..., 0)$, with 1 in the ith place.

First at all,

$$\mathcal{A}(\mathbf{f})(\mathbf{c}) = a_0(\mathbf{c}) \tag{14}$$

Then, for $i = 1, ..., n$, we have

$$D^{\mathbf{e}_i}\mathcal{A}(\mathbf{f})(\mathbf{c}) = D^{\mathbf{e}_i}a_0(\mathbf{c}) + a_{\mathbf{e}_i}(\mathbf{c}) \tag{15}$$

and we can get $D^{\mathbf{e}_i}a_{\mathbf{0}}(\mathbf{c})$ from the solution $\mathbf{a}_i = (D^{\mathbf{e}_i}a_{\eta}(\mathbf{c}))_{0\leq|\eta|\leq 1}$ of

$$FW(\mathbf{c})F^{T}(\,\mathbf{a}_i) = F(D^{\mathbf{e}_i}W(\mathbf{c}))\,(\mathbf{f} - F^{T}\mathbf{a}) \tag{16}$$

3.2 The Star of nodes at a point $\mathbf{c} \in \overline{\Omega}$

As it is well know, in working with the normal equation (13) and all related equations, one can consider only those *nodes* $\mathbf{x}_{\alpha}$ such that $\mathcal{W}_{\alpha}(\mathbf{c}) \neq 0$, that is, the *star* $\mathcal{ST}(\mathbf{c})$. If $\mathcal{ST}(\mathbf{c}) = \{\alpha_1, ..., \alpha_k\}$, matrices F, W and $\mathbf{f}$ can be considered as

$$F = \begin{pmatrix} p_0(\mathbf{x}_{\alpha_1}) & p_0(\mathbf{x}_{\alpha_2}) & \cdots & p_0(\mathbf{x}_{\alpha_k}) \\ p_1(\mathbf{x}_{\alpha_1}) & p_1(\mathbf{x}_{\alpha_2}) & \cdots & p_1(\mathbf{x}_{\alpha_k}) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_{\alpha_1}) & p_m(\mathbf{x}_{\alpha_2}) & \cdots & p_m(\mathbf{x}_{\alpha_k}) \end{pmatrix}$$

$$W = \begin{pmatrix} \mathcal{W}_{\alpha_1}(\mathbf{c}) & 0 & \cdots & 0 \\ 0 & \mathcal{W}_{\alpha_2}(\mathbf{c}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_{\alpha_k}(\mathbf{c}) \end{pmatrix}$$

$$\mathbf{f} = (f_{\alpha_1}, f_{\alpha_2}, ..., f_{\alpha_k})$$

and so on.

The *size* of the *star* $\mathcal{ST}(\mathbf{c})$ is defined by the number

$$h(\mathcal{ST}(\mathbf{c})) = \max\{d_{\alpha_1}, ..., d_{\alpha_k}\}$$

In all of this section $\mathbf{c} \in \overline{\Omega}$ is a fixed point and $\mathcal{ST}(\mathbf{c})$ is the *star* at c. In order to gain clarity, we shall drop subscript α from the weight functions and *nodes* in the *star*. Then, for $i = 1, ..., k$, $\mathcal{W}_i$ means $\mathcal{W}_{\alpha_i}$, $\mathbf{x}_i$ means $\mathbf{x}_{\alpha_i}$, $x_{i,j}$ is the jth coordinate of $\mathbf{x}_{\alpha_i}$, etc. It will also be useful to introduce a linear change of coordinates by the formula $\mathbf{y} = \mathbf{x} - \mathbf{c}$. We also set $h_c = h(\mathcal{ST}(\mathbf{c}))$.

A fundamental result is:

Theorem 3.2.1 *There exists a computable number $CN_1(\mathcal{ST}(\mathbf{c}))$ and constant C_1, $C_1 = C(n, k, CN_1(\mathcal{ST}(\mathbf{c})))$, such that*

$$|a_\eta| \leq C_q h_c^{-|\eta|} ||V||, \qquad 0 \leq |\eta| \leq 1 \tag{17}$$

where $V \in \mathbb{R}^k$, and $\mathbf{a} = (a_\eta)_{0 \leq |\eta| \leq 1}$ is the solution of

$$FWF^T\mathbf{a} = FV \tag{18}$$

The *condition number* $CN_1(\mathcal{ST}(\mathbf{c}))$ is a geometrical measure of the quality of the *star* $\mathcal{ST}(\mathbf{c})$ and it is the fundamental ingredient in the error estimate of moving least square approximations. This result is used to obtain error estimates for MLSQ approximations in the space of differentiable functions. We can apply this theorem 3.2.1 in order to obtain, for example, estimates of $D^\eta \phi_\alpha(\mathbf{c})$, $0 \leq |\eta| \leq 1$, $\alpha = 1, ..., N$.

As we have remarked before, only data at *nodes* of the *star* participate in calculations.

Theorem 3.2.2 *Let $C_G > 0$ as in (H4). Then, there exists a constant $\widetilde{C}_1 = \widetilde{C}_1(n, k, C_G, CN_1(\mathcal{ST}(\mathbf{c})))$ such that*

$$|D^\eta \phi_\alpha(\mathbf{c})| \leq \frac{\widetilde{C}_1}{h_c}, \quad 0 \leq |\eta| \leq 1, \; \alpha = 1, ..., N$$

4 ERROR ESTIMATES FOR INTERPOLANTS

The *conditions numbers* CN_q, $q = 1, 2$, play a fundamental role in obtaining error estimates for MLSQ interpolants in the space of differentiable functions[7]. Theorem 3.2.2 can be used to obtain error estimates in the context of Sovolev spaces.

Let $u \in W^{2,q}(\Omega)$, $q \in [1, \infty]$. We assume

(H5) $2q > n$ if $q > 1$, or $n \leq 2$ if $q = 1$.

Then by the Sobolev embedding theorem, $u \in C(\overline{\Omega})$ and it is meaningful to use pointwise values of $u(x)$ and the MLS approximation of u is defined. By (11), the approximation is defined by the formula

$$\widehat{u}(\mathbf{x}) = \sum_{\alpha=1}^{N} u(\mathbf{x}_\alpha) \cdot \phi_\alpha(\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega}$$

Theorem 4.0.3 *Assume (H1), (H2), (H3), (H4) and (H5). Then, there exists a constant* $c = c(n, \widetilde{C}_1)$ *such that, for any* $u \in W^{2,q}(\Omega)$, *we have the error estimates*

$$|| \, u - \widehat{u} \, ||_{W^{l,q}(\Omega)} \leq c d^{2-l} | \, u \, |_{W^{2,q}(\Omega)}, \qquad l = 0, ..., 2$$

where d *is as in (8).*

This result was proved by W. Han and X. Meng in the context of RKPM approximation method (cf. Han[13], Section 4.3). The proof of the theorem above follows exactly along the same lines and we shall omit the proof. See also Duarte[15,16] for other error estimations.

Given the following variational problem: find $u \in V \subset H^1(\Omega)$ such that

$$B(u, v) = L(v) \qquad \forall v \in V,$$

where B is a bilinear, continuous and coercive on V and L is a linear continuous operator, we can use the MLS method to define Galerkin approximation in the following way:

Assuming that $\varphi_\alpha \in V$, $\alpha = 1, ..., N$, let $V_N = span\{\phi_1, ..., \phi_N\}$. Therefore we can define the Galerkin approximation $\widehat{u} \in V_d$ of the real solution u as

$$\widehat{u}(\mathbf{x}) = \sum_{\alpha=1}^{N} u_\alpha \phi_\alpha(\mathbf{x})$$

where $u_1, ..., u_N$ is the solution of the following system

$$\sum_{\beta=1}^{N} B(\phi_\alpha, \phi_\beta) \, u_\beta = L(\phi_\alpha), \qquad 1 \leq \alpha \leq N$$

If $u \in H^2(\Omega)$ and assumption G above holds, then from Céa's lemma[14] and Theorem 4.0.3 we have the following error estimate:

$$||u - \widehat{u}||_V \leq \frac{K}{\lambda} \min_{v \in V_N} || u - v ||_V \leq \frac{K}{\lambda} || u - \widehat{u} || \leq C\, d^2 |\, u\, |_{H^2(\Omega)}$$

This estimate does not include Dirichlet boundary value problems. Derivation of rigorous error estimates in the latter case is much more difficult since in general $\widehat{u}$ does not satisfies Dirichlet condition.

5 GOOD QUALITY POINT SETS FOR MLSQ

Recently there has been a rapidly growing body of literature concerning the generation of good point sets for the use of meshless methods in solving PDEs[9,10,11,12]. Similar to mesh generation problems, meshless method also induce challenging problems. Partition of Unity Method (PUM) or MLS based methods are based on the definition of overlapping patches covering the domain and they raise an important geometric problem that is how to locate these patches so that the numerical problem is simulated accurately. Theorems 3.2.2 and 4.0.3 seem to hint that the *condition number* $CN_1(\mathcal{ST}(\mathbf{c}))$, $\mathbf{c} \in \overline{\Omega}$, could be used as a good measure of the quality of the distribution of *nodes* and patches. We further explore this point in this section.

Let $\mathbf{c} \in \overline{\Omega}$ be fixed. Writing $\mathbf{y} = \mathbf{x} - \mathbf{c}$, we look at the coordinates function y_i, $i = 1, ..., n$, as random variables over the probability space $\{(\mathbf{x}_\alpha), \mathcal{W}_\alpha)_{\alpha \in \mathcal{ST}(\mathbf{c})}$, where $\mathcal{W}_\alpha = \mathcal{W}_\alpha(\mathbf{c})$. For the sake of simplifying notation we shall also write $y_{i,\alpha} = y_i(\mathbf{x}_\alpha)$.

The matrix associated to the normal equation (10) can be written

$$A = \begin{pmatrix} 1 & \sum \mathcal{W}_\alpha \cdot y_{1,\alpha} & \cdots & \sum \mathcal{W}_\alpha \cdot y_{n,\alpha} \\ \sum \mathcal{W}_\alpha \cdot y_{1,\alpha} & \sum \mathcal{W}_\alpha \cdot y_{1,\alpha}^2 & \cdots & \sum \mathcal{W}_\alpha \cdot y_{1,\alpha}\, y_{n,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \sum \mathcal{W}_\alpha \cdot y_{n,\alpha} & \sum \mathcal{W}_\alpha \cdot y_{n,\alpha}\, y_{1,\alpha} & \cdots & \sum \mathcal{W}_\alpha \cdot y_{n,\alpha}^2 \end{pmatrix}$$

By Gauss procedure and recasting all elements in the matrix by elementary statistic theory, matrix A is transformed to

$$\widetilde{A} = \begin{pmatrix} 1 & \cdots \\ 0 & \overline{A}_{11} \end{pmatrix}$$

where $\overline{A}_{11}$ is the matrix of central moments of the random variables (y_i) :

$$\begin{pmatrix} \sum \mathcal{W}_\alpha \cdot (y_{1,\alpha} - \overline{y}_1)^2 & \cdots & \sum \mathcal{W}_\alpha \cdot (y_{1,\alpha} - \overline{y}_1)(y_{n,\alpha} - \overline{y}_n) \\ \vdots & \ddots & \vdots \\ \sum \mathcal{W}_\alpha \cdot (y_{1,\alpha} - \overline{y}_1)(y_{n,\alpha} - \overline{y}_n) & \cdots & \sum \mathcal{W}_\alpha \cdot (y_{n,\alpha} - \overline{y}_n)^2 \end{pmatrix}$$

If h_c is the *size* of *star* $\mathcal{ST}(\mathbf{c})$, then we normalize matrix $\overline{A}_{11}$ by setting

$$C = \left(\frac{1}{h_c^2}\right)\overline{A}_{11}$$

The 1-condition number is defined in Zuppa[7] by formula

$$CN_1(\mathcal{ST}(\mathbf{c})) = ||\, C^{-1}||$$

It is clear that $CN_1(\mathcal{ST}(\mathbf{c}))$ it is a geometrical measure of the quality of the distribution of nodes $\{\mathbf{x}_\alpha\}_{\alpha\in\mathcal{ST}(\mathbf{c})}$ and values $\{\mathcal{W}_\alpha(\mathbf{c})\}_{\alpha\in\mathcal{ST}(\mathbf{c})}$ around point $\mathbf{c}$.

Remark 5.0.4 *To be strictly invariant by dilations, functions* $\{\psi_\alpha\}$ *must be obtained at different size grids by scaling a fixed function, a standard procedure in partition of unity method.*

A moderate oscillatory behaviour of $CN_1(\mathcal{ST}(\mathbf{c}))$ over domain Ω and increasing values when $\mathbf{c}$ approach the boundary of Ω are expected. This is because near the boundary the *stars* tend to be of strongly unsymmetrical shape, with the center of mass shifted towards the inside of the domains.

To get a better idea of what we should expect we have made a numerical experiment. Tests with both random and uniformly spaced nodes were performed in $\overline{\Omega} = [0,1]^2$. In the former case, nodes were generated by adding a random perturbation of value $0.30h$ to a uniform grid with $h-$spacing with $h = 0.0625$. Following the scheme of section 2, the *cloud* ω_α will be the open ball $B_{d_\alpha}(\mathbf{x}_\alpha)$ such that 1 is verified and $d_\alpha = rh$. The experiments was made with the choice $r = 1.4$. The $C^\infty-$function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(\mathbf{x}) = \begin{cases} \exp(1)\cdot\exp\left(\frac{1}{||x||^2-1}\right), & \text{if } ||\,\mathbf{x}\,|| < 1 \\ 0, \qquad \text{otherwise} \end{cases}$$

was used for building the partition of unity and, for $\alpha = 1, ..., N$, the function ψ_α with support in $B_{d_\alpha}(\mathbf{x}_\alpha)$ was defined by formula

$$\psi_\alpha(\mathbf{x}) = \varphi\left(\frac{\mathbf{x}-\mathbf{x}_\alpha}{d_\alpha}\right)$$

The MLS approximation of function $f(x,y) = \cos(2\pi x)\sin(2\pi y)$ was considered over $\overline{\Omega}$ and the maximal absolute error in the function and its derivatives are displayed in the table bellow for uniform and random grids.

Grid	$\max\|u-\widehat{u}\|$	$\max\|u_x-\widehat{u}_x\|$	$\max\|u_y-\widehat{u}_y\|$	$\max CN_1$	$mean(CN_1)$
Uniform	4.45e-002	1.17e+000	5.62e-001	1.46e+001	7.17e+000
Random	5.76e-002	2.16e+000	2.04e+000	4.48e+002	2.23e+001

It is clear that works dealing with the problem of how to locate the nodes and their patches might be improved with the inclusion in criteria that define a good point set, the minimization of some function like

$$\sum_{\alpha} CN_1(\mathcal{ST}(\mathbf{x}_\alpha))$$

or a similar expression. This issue remains an object of further study.

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