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A STRAIGHTFORWARD APPROACH TO SOLVE ORDINARY NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. *The analytical solution of nonlinear differential systems is addressed. The approach consists in algebraic series in the time variable that leads to elementary recurrence algorithms. This is an alternative to standard techniques for numerical integration and it ensures the theoretical exactness of the response. Here it is shown that the systematic extension to nonlinear problems leads to a very convenient numerical integration algorithm without any approximation or truncation in each time step as is the case with the standard integration schemes (Newmark, α-method, Wilson-θ, Runge-Kutta).*
Most of usual nonlinearities may be represented by an algebraic series. The calculation of the derivatives is immediate. Factors of the different powers are equated giving rise to the ***algebraic recurrence algorithm.*** *Since a unitary domain is convenient, a nondimensionalization of the variable t is done using T, an interval of interest —$\tau = t/T$.*
When problems of numerical divergence appear, steps of appropriate duration T are used.
Several examples complete this study. They are: a) projectile motion, b) N bodies with gravitational attraction, c) Lorenz equations, d) Duffing oscillator and, e) a strong nonlinear oscillator. The results are given in plots, state variables vs. time, phase plots and Poincaré maps. Neither divergence nor numerical damping was found for the chosen values of T.

1 INTRODUCTION

The computer has been for the last fifty years an extraordinary tool to tackle problems absolutely unthinkable to the pioneers of the numerical calculus. Furthermore the personal computers widely used in the last decades have made the numeric resource an extended tool that the user generally employs without knowing the basis of the algorithm. On the other hand an inexpert user usually accepts without further discussion the outcomes without attempting other alternative other that his commercial software. Perhaps with a small computational effort, additional tools could be available that allow to overcome, for instance, divergences in the numerical integration when standard methods are employed.

Following this argument and taking advantage of the capability of the present computers, we here propose a methodology to integrate ordinary differential systems, linear and nonlinear, with constant or variable coefficients with initial or boundary conditions. As is known, in certain cases[1,2] the classical methods (Newmark, α-method, Wilson-θ, Runge-Kutta) show a divergent behavior. Instead the present technique without truncation and used as a time integration scheme gives no place to either divergence or numerical damping and ensures the arbitrary precision of the results. Here, however, the study of the numerical problems is beyond the pretense of the authors.

The (simple) motivation on which the methodology is based is that if the solution of the problem is continuous in a certain domain, then it may, in general, be represented with uniform convergence with algebraic series. When dealing with linear problems Frobenious type methods are founded in this feature. In the present method no indicial equation is analyzed and the case of no regularity is overcome with expansions around an appropriate center.

On this basis, and introducing a set of systematized simplifications, many nonlinearities may be represented in a simple fashion by an algebraic series. The calculation of the derivatives is immediate. Factors of the different powers are equated giving rise to the ***algebraic recurrence algorithm.***

Since it is convenient to use a unitary domain, a nondimensionalization of the independent variable *t* is done using *T,* an interval of interest —$\tau = t/T$. When problems of numerical divergence appear, steps of appropriate duration *T* are used.

Several examples complete this study. They are a) projectile motion, b) *N* bodies with gravitational attraction, c) Lorenz equations, d) Duffing oscillator and, e) a strong nonlinear oscillator. The results are given in plots, state variables vs. time, phase plots and Poincaré maps. Neither divergence nor numerical damping was found for the chosen values of *T.* To fix ideas Bessel´s equation is solved in Section 3.

2 GENERAL APPROACH

2.1 Analytical function

Let $F = \hat{F}(x, y, z)$ be an analytical function in three dimensions over a cubic domain *R*. where in turn *x, y* and *z* are functions of a parameter *τ*, $x = x(\tau), y = y(\tau), z = z(\tau)$ with $0 \le \tau \le 1$.

That is, $F = \hat{F}(x(\tau), y(\tau), z(\tau)) = F(\tau)$. A Taylor expansion yields

$$F = \sum_{p=0} \sum_{q=0} \sum_{r=0} \alpha_{pqr} x^p y^q z^r \tag{1}$$

in which the α_{pqr}'s are known coefficients. N.B. When the upper limit of the summations is not written, it is understood that theoretically it is infinity; however in practice, this limit is fixed by the desired precision. Let us accept that the functions $x(\tau), y(\tau), z(\tau)$ and their powers may be expanded in series of τ, and introduce the following notation

$$\left[x^m(\tau)\right] = \sum_{i=0} A_{mi}\tau^i; \quad \left[y^m(\tau)\right] = \sum_{i=0} B_{mi}\tau^i; \quad \left[z^m(\tau)\right] = \sum_{i=0} C_{mi}\tau^i \tag{2}$$

where *m* is a positive integer power. We now impose that

$$[F(\tau)] = \sum_{n=0} \phi_n \tau^n \tag{3}$$

In order to calculate ϕ_n, it is necessary to first determinate the following successive products. Introducing

$$\left[x^p(\tau) y^q(\tau)\right] = \sum_{i=0} a_{pqi}\tau^i \tag{4}$$

it is easily observed that, in virtue of the product of series, one obtains

$$a_{pqi} = \sum_{s=0}^{i} A_{ps} B_{q(i-s)} = \sum_{s=0}^{i} B_{qs} A_{p(i-s)} \tag{5}$$

Then, if

$$\left[x^p(\tau) y^q(\tau) z^r(\tau)\right] = \sum_{i=0} b_{pqri}\tau^i \tag{6}$$

a way to calculate b_{pqri} is, analogously to (5), the following expression

$$b_{pqri} = \sum_{s=0}^{i} a_{pqs} C_{r(i-s)} \tag{7}$$

In this manner, after combining expressions (1), (3) and (7), we find

$$\phi_n = \sum_{p=0} \sum_{q=0} \sum_{r=0} \alpha_{pqr} b_{pqrn} \tag{8}$$

On the other hand, and based on the product of series and

$$x^m(\tau) = x^{m-1}(\tau)x(\tau); \quad y^m(\tau) = y^{m-1}(\tau)y(\tau); \quad z^m(\tau) = z^{m-1}(\tau)z(\tau) \tag{9}$$

the following relationships are obtained

$$A_{mi} = \sum_{s=0}^{i} A_{(m-1)s} A_{1(i-s)}; \quad B_{mi} = \sum_{s=0}^{i} B_{(m-1)s} B_{1(i-s)}; \quad C_{mi} = \sum_{s=0}^{i} C_{(m-1)s} C_{1(i-s)} \qquad m = 2,3,\cdots \tag{10}$$

In equations (10) identities are obtained when m=1 as may be inferred from (2), and their values are

$$A_{00} = B_{00} = C_{00} = 1 \quad but \quad A_{0i} = B_{0i} = C_{0i} = 0 \tag{11}$$

Then the coefficients A_{mi}, B_{mi}, and C_{mi} $(m \geq 2)$ are found starting from the A_{1i}, B_{1i} and C_{1i} supposedly known. The shown systematizations for a 3D analytical function are general and stand for other dimensions.

2.2 **Derivatives of *F(τ)***

For the sake of simplicity the following notation is introduced

$$(\bullet)^{(k)} = \frac{d^k(\bullet)}{d\tau^k} \quad (k = 1,2,\cdots) \qquad \varphi_{kj} \equiv (j+1)(j+2)\cdots(j+k) \quad (k, j \textit{ positive integers})$$

Now we are able to write

$$\left[F^{(k)}(\tau)\right] = \sum_{n=0} \varphi_{kn} \phi_{(n+k)} \tau^n \tag{12}$$

2.3 Calculation of coefficients of a function product

Let $f(\tau)$ and $g(\tau)$ be known functions analytic in $\tau = 0$ and defined in $\{I : 0 \leq \tau \leq 1\}$. Then

$$[f(\tau)] = \sum_{i=0} A_i \tau^i ; \quad [g(\tau)] = \sum_{i=0} B_i \tau^i \tag{13}$$

and the product

$$[h(\tau)] = [f(\tau) g(\tau)] = \sum_{i=0} C_i \tau^i \tag{14}$$

also analytic in $\tau = 0$ and defined in *I*. It is straightforward to find that

$$\begin{aligned} C_i &= \sum_{s=0}^{i} A_s B_{(i-s)} = \sum_{s=0}^{i} B_s A_{(i-s)} \\ &(i = 0,1,2,\cdots) \end{aligned} \tag{15}$$

This expression includes the case $f(\tau) = [g(\tau)]^p$ with *p* integer. In order to calculate $[g(\tau)]^p$, $[g(\tau)]^{p-1}$ is needed and so on, successively applying expression (15).

2.4 Calculation of the coefficients of a rational function

Let $f(\tau)$ and $g(\tau)$ be known analytical functions at $\tau = 0$ and defined in $\{I : 0 \le \tau \le 1\}$. Furthermore $g(\tau) \ne 0$, $\tau \in I$. Then if $h(\tau)$ is a rational function in *I*,

$$[h(\tau)] = \left[\frac{f(\tau)}{g(\tau)} \right] = \sum_{i=0} D_i \tau^i \tag{16}$$

To calculate the coefficients D_i the following consistence condition is imposed

$$[f(\tau)] = [h(\tau) g(\tau)] \tag{17}$$

After using expansions (13) and (16) and (15) we observe that

$$A_i = \sum_{s=0}^{i} D_s B_{(i-s)} = \sum_{s=0}^{i} B_s D_{(i-s)} \tag{18}$$

From the right member of expression (18) one may write

$$A_i = B_0 D_i + \sum_{s=1}^{i} B_s D_{(i-s)} \tag{19}$$

Then the unknowns may be found as

$$D_i = \frac{A_i - \sum_{s=0}^{i} B_s D_{(i-s)}}{B_0} \tag{20}$$

Coefficients D_i are to be found sequentially. $B_0 = g(0) \neq 0$ by hypothesis.

2.5 Fractional power of an analytical function

Let $f(\tau)$ (expression (13)) be a known function defined in I and analytic in $\tau = 0$ and, assuming n a positive integer,

$$[h(\tau)] = \left[f(\tau)^{1/n} \right] \equiv \left[\sqrt[n]{f(\tau)} \right] = \sum_i P_{1i} \tau^i \tag{21}$$

The algebraic consistence requires that

$$\sum_i A_i \tau^i = [f(\tau)] = \left[h(\tau)^n \right] = \sum_i P_{ni} \tau^i \tag{22}$$

After equating, one finds that

$$P_{ni} = A_i \tag{23}$$

The unknowns P_{1i} (equation (21)) may be obtained by recurrence. It is not difficult to infer that if we introduce

$$Z_{ri} = \sum_{s=1}^{i-1} P_{rs} P_{1(i-s)}; \quad r = 1,2,\cdots,(n-1) \tag{24}$$

$$D_{n0} = P_{10}^{n-1} + \sum_{j=1}^{n-1} P_{10}^{j-1} P_{(n-j)0} \tag{25}$$

then

$$P_{1i} = \frac{A_i - \sum_{j=1}^{n-1} P_{10}^{j-1} Z_{(n-j)i}}{D_{n0}} \tag{26}$$

Then the coefficients P_{1i} may be found sequentially for $i \geq 1$. N.B. In order to obtain good approximations the number of terms of the series should be large (e.g. if $n = 5 \Rightarrow M \geq 3000$).

3 SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

3.1 A special situation

Before the statement of the power series for nonlinear problems, aim of this paper, we first show a problem that usually appears in the resolution of differential equations. Let us analyze, for instance, a second order differential equation in $v = v(\tau)$ $(\{I : 0 \le \tau \le 1\})$ where we find a term of the form $f(\tau)v''(\tau)$, but $f(\tau)$ assumes a null value for a certain $\tau_0 \in I$. Then both $f(\tau)$ and $v''(\tau)$ have to be expanded in power with center in $\tau_1 \ne \tau_0$, $\tau_1 \in I$ where $f(\tau_1) \ne 0$. Then

$$[f(\tau)] = \sum_{i=0} \alpha_i (\tau - \tau_1)^i; \quad [v(\tau)] = \sum_{i=0} A_i (\tau - \tau_1)^i \tag{27}$$

in which α_i are known. Consequently (see Section 2.2)

$$[v''(\tau)] = \sum_i \varphi_{2i} A_{(i+2)} (\tau - \tau_1)^i \tag{28}$$

From Section 2.3

$$\begin{aligned} &[f(\tau)v''(\tau)] = \sum_{i=0} B_i (\tau - \tau_1)^i && (a) \\ &B_i = \sum_{s=0}^{i} \varphi_{2s} A_{(s+2)} \alpha_{(i-s)} && (b) \end{aligned} \tag{29}$$

Since we are dealing with a second order equation, $A_{(i+2)}$ are required. Then we rewrite expression (29) as

$$B_i = \alpha_0 \varphi_{2i} A_{(i+2)} + \sum_{s=0}^{i-1} \varphi_{2s} A_{(s+2)} \alpha_{(i-s)} \tag{30}$$

It was assumed that $\alpha_0 = f(\tau_1) \ne 0$ and then it is possible to obtain $A_{(i+2)}$. Let us illustrate this topic with the Bessel's equation

$$x^2 v'' + x v' + (x^2 - \nu^2) v = 0 \tag{31}$$

where $v = v(x)$, $0 \le x \le T$ and T is an interval of interest for the solution. In order to ensure —in general— the convergence of the series, we introduce $\tau \equiv x/T$ $(0 \le \tau \le 1)$. If we denote $\overline{(\bullet)} \equiv d(\bullet)/d\tau$, equation (31) may be rewritten with $v = v(\tau)$ as

$$\tau^2 \overline{\overline{v}} + \tau \overline{v} + (T^2 \tau^2 - \nu^2) v = 0 \tag{32}$$

Consequently the function that premultiplies $\bar{\bar{v}}$ is null at $\tau = 0$. Then we may expand, for instance, around $\tau_1 = 1\ (\tau_1^2 \neq 0)$ as follows

$$
\begin{aligned}
&[\tau] = \sum_i \alpha_i (\tau - 1)^i \quad \Rightarrow \quad \alpha_0 = \alpha_1 = 1; \quad \alpha_i = 0\ (i \neq 0,1) \\
&\left[\tau^2\right] = \sum_i \beta_i (\tau - 1)^i \quad \Rightarrow \quad \beta_0 = 1;\ \beta_1 = 2;\ \beta_2 = 1
\end{aligned}
\tag{33}
$$

that may be found with expressions (15). Furthermore

$$
[v] = \sum_i A_i (\tau - 1)^i; \qquad [\bar{v}] = \sum_i \varphi_{1i} A_{(i+1)} (\tau - 1)^i; \quad \left[\bar{\bar{v}}\right] = \sum_i \varphi_{2i} A_{(i+2)} (\tau - 1)^i \tag{34}
$$

$$
\left[\tau^2 \bar{\bar{v}}\right] = \sum_i B_i^* (\tau - 1)^i; \qquad [\tau\bar{v}] = \sum_i C_i (\tau - 1)^i; \quad \left[\tau^2 v\right] = \sum_i D_i (\tau - 1)^i \tag{35}
$$

with which, after using (20), we may write

$$
\begin{aligned}
&B_i^* = \varphi_{2i} \beta_0 A_{(i+2)} + B_i \quad \Rightarrow \quad B_i = \sum_{s=1}^{i} \varphi_{2(i-s)} \beta_s A_{(i-s+2)} \\
&C_i = \sum_{s=0}^{i} \varphi_{1(i-s)} \alpha_s A_{(i-s+1)}; \quad D_i = \sum_{s=0}^{i} \beta_s A_{(i-s)}
\end{aligned}
\tag{36}
$$

The recurrence for the differential equation (32) arises from equating the series coefficients for each *i-th* power of $(\tau - 1)$ and yields

$$
B_i^* + C_i + T^2 D_i - \nu^2 A_i = 0 \tag{37}
$$

or

$$
A_{(i+2)} = \frac{\nu^2 A_i - (B_i + C_i + T^2 D_i)}{\varphi_{2i}} \tag{38}
$$

Since A_0 and A_1 are free coefficients the two independent solutions are found by imposing successively $A_0 = 1,\ A_1 = 0$ and $A_0 = 0,\ A_1 = 1$. It should be noted that, evidently, both solutions are not coincident with the classical solutions J_ν (first kind) nor Y_ν (second kind) but their combination, after the substitution of the boundary conditions, leads to the same result. Recall that in case of integer ν the classical Y_ν includes a logarithmic part that in this methodology is replaced by an expansion around $\tau_1 = 1$.

Finally the situation including the term of the form $f(\tau)v''(\tau)$ may be tackled as seen above but also it might be addressed by first dividing all the differential equation by $f(\tau)$. In this case one would encounter rational functions non analytic in $\tau = 0$. This approach would lead to a heavier algebraic work but the solution would be as efficient as the previous one.

3.2 Nonlinear differential equations

In the next item various examples of nonlinear differential equation systems will be solved using the present methodology. However, and with the aim of showing a wider scope, we will now illustrate in detail the Duffing equation modeling, for instance, the forced vibration of beams. An alternative to use the method as a time integration technique will be shown. The exact solution will be found in an arbitrary time interval. This time interval is in turn subdivided in subdomains of duration *T*. Then, the Duffing equation writes

$$\begin{cases} \ddot{u} + P^{*}\dot{u} + Q^{*}u^{3} = B^{*}f(t) \\ IC: u(0) = U_0 \text{ and } \dot{u}(0) = V_0^{*} \end{cases} \tag{39}$$

where *t* is the time, $u = u(t)\ (t \geq 0)$ and P^{*}, Q^{*} and B^{*} are parameters that derive from the physical model. $f = f(t)$ is the load that excites the s.d.o.f. system. For certain range of parameters the time response of the system is chaotic. Admitting that we are interested in the answer for large *t*, the direct approach in time series would lead to a lack of convergence. Thus the introduction of a nondimensional variable τ_p appears convenient:

$$t = T\left(\tau_p + p - 1\right) \tag{40}$$

where *T* is an interval of interest that is adjusted at will to obtain convergence; *p* is the arbitrary number of steps $(p = 1,2,\cdots)$. The domain of each τ_p is unitary which ensures the convergence, i.e.

$$(p-1) \leq t \leq pT \quad \Rightarrow \quad 0 \leq \tau_p \leq 1; \quad p = 1,2,\cdots \tag{41}$$

The expansions in each step will be in powers of τ_p with null center. Then for each step

$$\begin{aligned} u = u(t) = u\left[T\left(\tau_p + p - 1\right)\right] = v(\tau_p) = v_p(\tau_p) \\ f = f(t) = f\left[T\left(\tau_p + p - 1\right)\right] = f(\tau_p) = f_p(\tau_p) \end{aligned} \tag{42}$$

Equation (39) now writes, after introducing the notation $\overline{(\bullet)} \equiv d(\bullet)/d\tau_p$,

$$\begin{cases} \bar{\bar{v}}_p + P\bar{v}_p + Qv_p^{3} = Bf_p \\ IC: v(0) = U_0 \text{ and } \bar{v}(0) = V_0 \end{cases} \tag{43}$$

where

$$P = TP^*; Q = T^2 Q^*; B = T^2 B^*; V_0 = TV_0^* \tag{44}$$

Let us now propose the following expansions

$$\begin{aligned} &[v_p] = \sum_i A_{1i}^{\langle p \rangle} \tau_p^i; \quad [v_p^2] = \sum_i A_{2i}^{\langle p \rangle} \tau_p^i; \quad [v_p^3] = \sum_i A_{3i}^{\langle p \rangle} \tau_p^i; \\ &[\bar{v}_p] = \sum_i \varphi_{1i} A_{1(i+1)}^{\langle p \rangle} \tau_p^i; \qquad [\bar{\bar{v}}] = \sum_i \varphi_{2i} A_{1(i+2)}^{\langle p \rangle} \tau_p^i \end{aligned} \tag{45}$$

As mentioned in Section 2.3, expression (15), the following holds

$$\begin{aligned} A_{2i}^{\langle p \rangle} &= \sum_{s=0}^{i} A_{1s}^{\langle p \rangle} A_{1(i-s)}^{\langle p \rangle} \\ A_{3i}^{\langle p \rangle} &= \sum_{s=0}^{i} A_{2s}^{\langle p \rangle} A_{1(i-s)}^{\langle p \rangle} \end{aligned} \tag{46}$$

We assume

$$f(t) = \cos(\omega t) \tag{47}$$

being ω the circular forcing frequency. The function at each step is written as

$$\begin{aligned} &f_p(\tau_p) = \cos[\omega T(\tau_p + p - 1)] \\ &or \\ &f_p(\tau_p) = \cos[\omega T(p-1)]\cos(\omega T \tau_p) - \sin[\omega T(p-1)]\sin(\omega T \tau_p) \end{aligned} \tag{48}$$

The values of $\cos[\omega T(p-1)]$ and $\sin[\omega T(p-1)]$ are known for each step and after T has been chosen. Now if we denote $x_p = x(\tau_p) = \omega T \tau_p$, following Section 2.1 the next expansions hold

$$\begin{aligned} \cos x_p &= \sum_{k=0,2,4,\cdots} (-1)^{k/2} \frac{x_p^k}{k!} \equiv \sum_j \alpha_j x_p^j \\ \sin x_p &= \sum_{k=1,3,5,\cdots} (-1)^{(k-1)/2} \frac{x_p^k}{k!} \equiv \sum_j \beta_j x_p^j \end{aligned} \tag{49}$$

in which, as is well known,

$$\alpha_j = \begin{cases} 0 & if\ j\ odd \\ \dfrac{(-1)^{j/2}}{j!} & if\ j\ even \end{cases} \qquad \beta_j = \begin{cases} \dfrac{(-1)^{(j-1)/2}}{j!} & if\ j\ odd \\ 0 & if\ j\ even \end{cases}$$

$$j = 0,1,2,\cdots \tag{50}$$

In turn

$$\left[x^m(\tau_p)\right] = \sum_{i=0} B_{mi}^{\langle p\rangle} \tau_p^i \tag{51}$$

It is easily observed that

$$B_{0i}^{\langle p\rangle} = \begin{cases} 1 & if\ i=0 \\ 0 & if\ i \neq 0 \end{cases}; \qquad B_{1i}^{\langle p\rangle} = \begin{cases} \omega T & if\ i=0 \\ 0 & if\ i \neq 0 \end{cases} \tag{52}$$

As stated in Section 2.3 B_{mi} is calculated with expression (15) for $m \geq 2$. For this particular case —$x = \omega T \tau_p$— it is clear that

$$B_{mi}^{\langle p\rangle} = \begin{cases} (\omega T)^m & if\ i = m \\ 0 & if\ i \neq m \end{cases} \tag{53}$$

Consequently, if

$$\left[\cos(\omega T \tau_p)\right] = \sum_i C_i^{\langle p\rangle} \tau_p^i; \quad \left[\sin(\omega T \tau_p)\right] = \sum_i D_i^{\langle p\rangle} \tau_p^i \tag{54}$$

then

$$C_i^{\langle p\rangle} = \sum_{j=0} \alpha_j B_{ji}^{\langle p\rangle}; \quad D_i^{\langle p\rangle} = \sum_{j=0} \beta_j B_{ji}^{\langle p\rangle} \tag{55}$$

The recurrence condition that results from the statement of the differential equation (43) is, for each step,

$$\begin{aligned} &\varphi_{2i} A_{1(i+2)}^{\langle p\rangle} + P\varphi_{1i} A_{1(i+1)}^{\langle p\rangle} + Q A_{3i}^{\langle p\rangle} = \\ &\qquad B\left\{\cos\left[\omega T(p-1)\right)C_i^{\langle p\rangle}\right] - \sin[\omega T(p-1)]D_i^{\langle p\rangle}\right\} \end{aligned} \tag{56}$$

from where $A_{1(i+2)}^{\langle p\rangle}$ may be found. The free coefficients $A_{10}^{\langle p\rangle}$ and $A_{11}^{\langle p\rangle}$ are imposed with the initial conditions at each step. Obviously for p=1, $A_{10}^{\langle 1\rangle} = U_0$ and $A_{11}^{\langle 1\rangle} = V_0$ but for the successive steps, $p \geq 2$

$$A_{10}^{\langle p \rangle} = v_{(p-1)}(1)$$
$$A_{11}^{\langle p \rangle} = \bar{v}_{(p-1)}(1) \tag{57}$$

This algorithm constitutes a numerical integration tool that instead of the truncation of Taylor series and improvement of the efficiency (e.g. Runge-Kutta and variants), decidedly takes the complete analytical expansion, only limited by the equipment hardware and/or the desired output speed.

4 EXAMPLES

Next five cases of strongly nonlinear differential systems are briefly presented, all of them solved with the methodology above explained. After the statement of the problem, the recurrence is written and the numerical results given as trajectories, phase planes and Poincaré maps. In all cases the time variable was nondimensionalized as indicated in (40) of the previous Section, thought not necessarily in all cases $p > 1$.

4.1 Projectile motion

This is a three-dimensional problem of a projectile shot in the air, taking into account the air resistance and the wind action, both dependent of the height z at each instant. The governing system is[3]

$$\begin{aligned} x'' &= -\beta\left[x' - T^2 V_x(z)\right]e^{-z/h} \\ y'' &= -\beta\left[y' - T^2 V_y(z)\right]e^{-z/h} \\ z'' &= -\beta\left[z' - T^2 V_z(z)\right]e^{-z/h} - gT^2 \end{aligned} \tag{58}$$

The differential system (58) may be written in a compact statement as

$$x'' = -\beta U; \quad y'' = -\beta V; \quad z'' = -\beta W - GT^2 \tag{59}$$

Each function U, V, W are in turn expanded in a time power series with coefficients u_i, v_i, w_i respectively. After a rather lengthy algebraic manipulation one arrives the algebraic recurrence of the form (for more details see Filipich, Rosales and Buezas[4])

$$A_{k+2} = -\beta\frac{u_k}{\varphi_{2k}}; \; B_{k+2} = -\beta\frac{v_k}{\varphi_{2k}}; \; A_{k+2} = -\frac{\beta w_k + T^2 g\delta_{0k}}{\varphi_{2k}} \tag{60}$$

with $k = 0,1,2,\cdots$ and δ_{0k} are the Kronecker's deltas. The numerical example was carried out with the following data:

- Initial position $(x,y,z)=(0,0,0)$
- Initial velocity $(x',y',z')=(4,0,0.5)$
- Wind velocity $V=(1000z, 500z, 0)$
- $\beta=0.01$, $g=10$, $h=1000$, $T=0.01$, $M=N=20$

The resulting trajectory is shown in Figure 1.

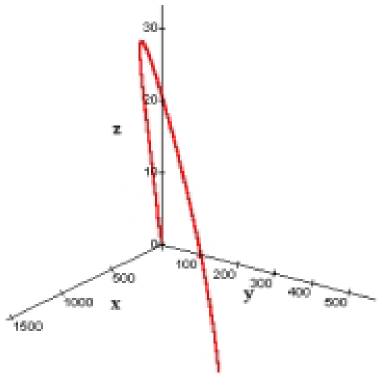


Figure 1. Example of Section 4.1. Projectile motion

4.2 Problem of the *N* bodies in a gravitational field

This section deals with the three-dimensional problem of N bodies attracted among them by a Newtonian gravitational force field. If $x_i = x_i(\tau);\ y_i = y_i(\tau);\ z_i = z_i(\tau)$ with $i = 1,2,\cdots$ are the Cartesian coordinates of each *i-th* body, then the scalar equations of motion are

$$\begin{cases} x_i'' = G\left(\displaystyle\sum_{k=i+1}^{N} \frac{m_k(x_k - x_i)}{R_{ik}^3}\right) - G\left(\displaystyle\sum_{k=1}^{i-1} \frac{m_k(x_i - x_k)}{R_{ki}^3}\right) \\ y_i'' = G\left(\displaystyle\sum_{k=i+1}^{N} \frac{m_k(y_k - y_i)}{R_{ik}^3}\right) - G\left(\displaystyle\sum_{k=1}^{i-1} \frac{m_k(y_i - y_k)}{R_{ki}^3}\right) \\ z_i'' = G\left(\displaystyle\sum_{k=i+1}^{N} \frac{m_k(z_k - z_i)}{R_{ik}^3}\right) - G\left(\displaystyle\sum_{k=1}^{i-1} \frac{m_k(z_i - z_k)}{R_{ki}^3}\right) \end{cases} \tag{61}$$

where $t = T(\tau_p + p - 1)$, m_i is the mass of the *i-th* body, $(\bullet)' \equiv d(\bullet)/d\tau$, $G =$ and

$$R_{ij} = \left|\vec{R}_{ij}\right| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

After a series of algebraic manipulations and denoting

$$[x_i(\tau)] = \sum_{k=0} A_{ik}\tau_p^k; \quad [y_i(\tau)] = \sum_{k=0} B_{ik}\tau_p^k; \quad [z_i(\tau)] = \sum_{k=0} C_{ik}\tau_p^k \tag{62}$$

the recurrence equations are obtained[5]

$$\begin{aligned} \varphi_{2k} A_{i(k+2)} &= G\left(\sum_{n=i+1}^{N} m_n X_{ink}\right) - G\left(\sum_{x=1}^{i-1} m_x X_{xik}\right) \\ \varphi_{2k} B_{i(k+2)} &= G\left(\sum_{n=i+1}^{N} m_n Y_{ink}\right) - G\left(\sum_{x=1}^{i-1} m_x Y_{xik}\right) \\ \varphi_{2k} C_{i(k+2)} &= G\left(\sum_{n=i+1}^{N} m_n Z_{ink}\right) - G\left(\sum_{x=1}^{i-1} m_x Z_{xik}\right) \end{aligned} \tag{63}$$

$$X_{rsk} = \sum_{p=0}^{k} \sigma_{rsp}(A_{s(k-p)} - A_{r(k-p)}); \quad Y_{rsk} = \sum_{p=0}^{k} \sigma_{rsp}(B_{s(k-p)} - B_{r(k-p)})$$

$$Z_{rsk} = \sum_{p=0}^{k} \sigma_{rsp}(C_{s(k-p)} - C_{r(k-p)}) \tag{64}$$

The numerical example was performed with the data in Table 1. The resulting trajectories are shown in Figure 2. The spheres indicate the initial position of the bodies.

Table 1. Example of Section 4.2. Data.

BODY	INITIAL POSITION	INITIAL VELOCITY	MASS
1	(0,0,0)	(0,0,0)	1
2	(3,0,0)	(0,0,0)	1
3	(3,4,0)	(0,0,0)	5
4	(1.5,1.5,0)	(-0.00125,0.0005,0.0005)	1

4.3 Lorenz equations

The well-known problem of Lorenz (see for instance Strogatz[6]), who discovered in 1963 the chaotic motions of a model of convection of rolls in the atmosphere, is herein solved with algebraic series. The behavior of this problem is such that very small changes in the initial conditions lead to different solutions but, as Lorenz showed, bounded in a three dimensional butterfly-shaped domain. The three-dimensional system derived by Lorenz is

$$\dot{u} = \sigma^* (v - u); \quad \dot{v} = r^* u - v - uw; \quad \dot{w} = uv - b^* w \tag{65}$$

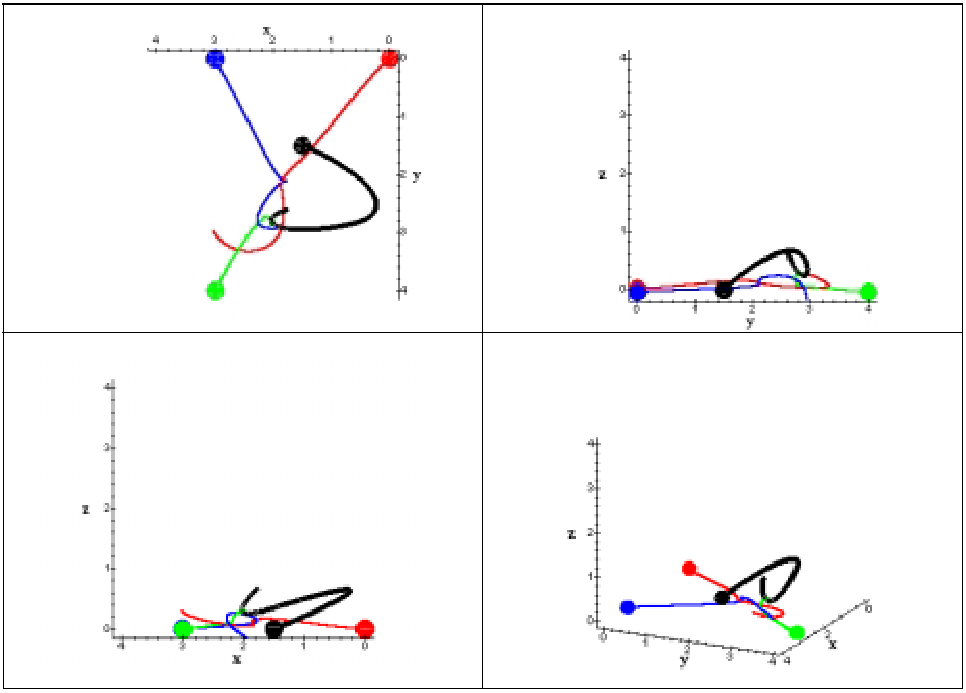


Figure 2. Example of 4 bodies in a gravitational field. Trajectories.

where $\sigma^*, r^*, b^* > 0$ are parameters; $u = \hat{u}(t)$; $v = \hat{v}(t)$; $w = \hat{w}(t)$. Here the variable $\tau = t / T$ is introduced. The following algebraic series are proposed

$$[u] = \sum_{i=0}^{N} A_i \tau^i; \quad [v] = \sum_{i=0}^{N} B_i \tau^i; \quad [w] = \sum_{i=0}^{N} C_i \tau^i; \tag{66}$$

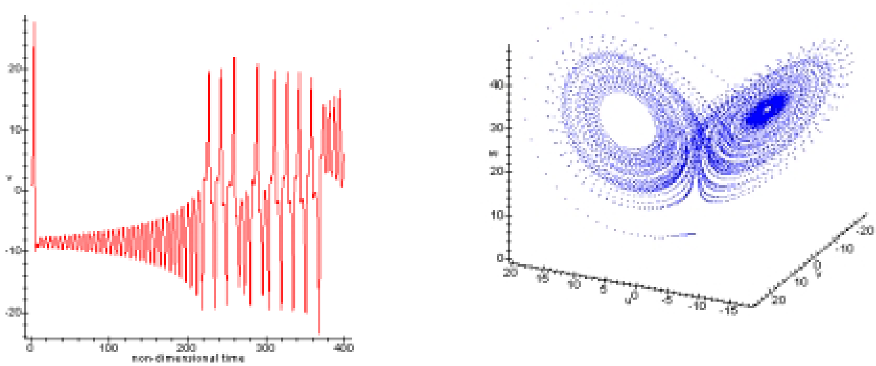
After replacement of these series in the differential system a recurrence system of equations is obtained as follows

$$A_{i+1} = \frac{\sigma}{\varphi_{1i}}(A_i - B_i); \quad B_{i+1} = \frac{1}{\varphi_{1i}}(rA_i - TB_i - TS_i); \quad C_{i+1} = \frac{1}{\varphi_{1i}}(TZ_i - bC_i) \tag{67}$$

with $\sigma = \sigma^* T; r = r^* T; b = b^* T$ and

$$S_i = \sum_{p=0}^{i} A_p C_{i-p}; \quad Z_i = \sum_{p=0}^{i} A_p B_{i-p} \tag{68}$$

The Lorenz problem was solved using the recurrence equation (67) for the data $u_0 = 0$, $v_0 = 1$, $w_0 = 0$. $T = 0.1$. $\sigma^* = 10, r^* = 28, b^* = 8/3$. N=20 and the trajectory of v is shown in Figure 3 a). The response is chaotic, as found by Lorenz in 1963. The phase plot of Figure 3 b) displays a bounded domain of the trajectory. This problem is known to be very sensitive to the initial conditions and poses a challenge to any numerical tool.



(a) (b)

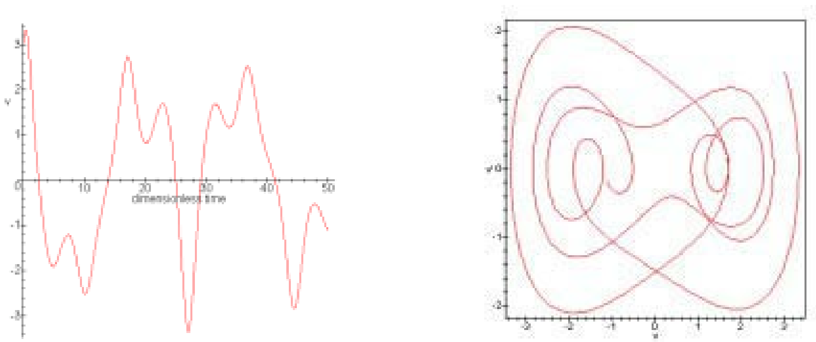
Figure 3. Lorenz attractor. $u_0 = 0$, $v_0 = 1$, $w_0 = 0$. $T = 0.1$. $\sigma^* = 10, r^* = 28, b^* = 8/3$. N=20. a) trajectory; b) strange attractor.

4.4 Duffing equation

This problem was already shown in Section 3.2, with the governing equation (39). A recurrence equation may be found as follows, and given that $a_{10} = U_0$ and $a_{11} = V_0$ the following recurrence expression yields

$$a_{1(i+2)} = \frac{B\alpha_i - P\varphi_i a_{1(i+1)} - Qa_{3i}}{\varphi_{2i}} \qquad (i = 0,1,2,\cdots,N-2) \tag{69}$$

with which the problem may be solved. Figure 4 a) shows the trajectory as a function of the dimensionless time (τ=t/T) with T=0.35 for the case $U_0 = 3$, $V_0^* = 4$, $P^* = 0.05$, $Q^* = 1$, $B^* = 7.5$, $f(t) = \cos t$ with N=20. Figure 4 b) depicts the corresponding phase plot. These results are identical to the ones reported in Thompson and Stewart[7]. It should be mentioned that this problem is extremely sensitive to the initial conditions. That is, is for instance one takes $U_0 = 3.1$, $V_0^* = 4.1$ the solution varies quite significantly from the one reported here, as Thompson and Stewart comment. The consequence of this feature is chaos. Hence the numerical behavior of the methodology is relevant to the reliability of the results. See more details in Filipich and Rosales[8]



(a) (b)

Figure 4. Solution of the forced Duffing equation. $U_0 = 3$, $V_0^* = 4$, $P^* = 0.05$, $Q^* = 1$, $B^* = 7.5$ .a) trajectory; b) phase plot.

4.5 Strongly nonlinear oscillator: $\ddot{x} + \omega^2 x + \varepsilon\,\dot{x}x^2 \sin 2t = 0$

This equation is related to rotor dynamics. Mahamud[9] studied it with an extended average theorem algorithm. It may represent the scalar part of a complex equation governing a damped non-linear system. Examples of this behavior appear also in robots and shells as reported by Mahamud[10]. The non-dimensionalized differential equation is written as

$$x'' + Ax + Bx'x^2 \sin 2T\tau = 0 \tag{70}$$

where $x = x(\tau_p)$, the non-dimensional time $(t - t_0) = T(\tau_p + p - 1)$, t_0 is the initial time, T is a time interval to be chosen and $(\cdot)' = \partial(\cdot)/\partial\tau$. The constants in the differential equations are $A = (\omega T)^2$ and $B = \varepsilon T$. The initial conditions are given by $x(0) = x_0$ and $x'(0) = x'_0$. Let us now state the algebraic series solution using the theory above developed. After some manipulation the differential equation (70) is written for each step as

$$(k+1)(k+2)a_{1(k+2)} + Aa_{1k} + B\lambda_k = 0 \qquad (k = 0,1,2,\ldots,N-2) \tag{71}$$

The initial conditions in each step give place to the starting values $a_{10} = x_{0p}$ and $a_{11} = x'_{0p}$. The algorithm is complete after the necessary *A.C.* of type (18) are stated. A numerical example was carried out setting $T = 0.125\text{sec}$, $\omega^2 = 0.306$, $\varepsilon = 5$, $t_0 = \pi$, $x_0 = -0.5$, $x'_0 = 0$. The number of terms in the series was taken $N = 50$. This example was solved by with an averaging method[9], as an extension to the approach for weakly non-linear systems. Mahmoud reports a figure of the trajectory of x(t) in a range [π,28.14] in which only two waves are observed. His results are close to the numerical solution (Runge-Kutta 4th. order) though a difference is noticeable in the plot. Figure 5 shows the trajectory found with the above-described algebraic recurrence plotted in the time domain along with the numerical solution found using the integration scheme forward Euler method, implemented in MAPLE V as default algorithm to solve differential equations

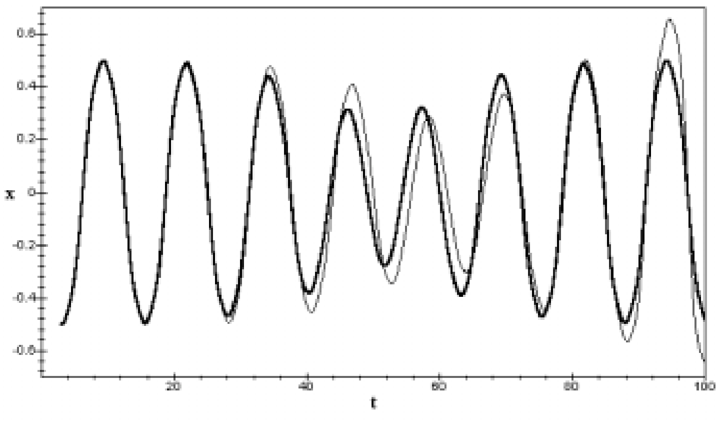
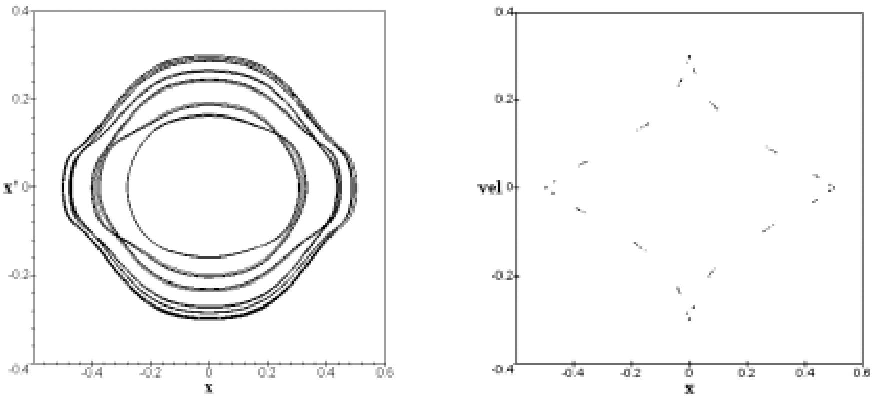


Figure 5. Trajectory $x(t)$. $T = 0.125\text{sec}$, $\omega^2 = 0.306$, $\varepsilon = 5$, $t_0 = \pi$, $x_0 = -0.5$, $x'_0 = 0$. $N = 50$.
(━━━) Algebraic series solution. (———) Numerical solution (forward Euler).

. It is seen that the latter numerical technique starts to diverge before the 30 sec. Actually the authors found long-time trajectories without divergence or damping (e.g. in the range [π,375]). The corresponding phase diagram is depicted in Figure 6 a. As may be seen both diagrams exhibit the characteristics of what might be a quasiperiodic motion (a hint of a modulation in the trajectory is seen in Figure 5 and in Filipich and Rosales[11]). The Poincaré map (Figure 6 b) was found for this example using a sampling time equal to π. It shows a diamond shaped array of points. However, although the total time was of 750 seconds, no closed figure is obtained. The partial segments arise from the transient behavior and it is concluded that the motion is periodic (a multiperiod response) and not quasiperiodic.



(a) (b)

Figure 6. a) Phase diagram. $N = 50$. $T = 0.125\text{sec}$, $\omega^2 = 0.306$, $\varepsilon = 5$, $x_0 = -0.5$, $x'_0 = 0$, $t_0 = \pi$. Time of experiment: 375 sec. b) Poincaré map. Time of experiment=750 sec. Sampling time: π. Approximately 240 points.

5 CONCLUSIONS

A methodology to find the analytical solution of non-linear differential equations has been presented. The technique makes use of algebraic series and simply recurrences are derived to solve the problem under study. Five illustrative examples are briefly shown: a projectile motion, n-bodies with gravitational attraction, Lorenz equations, Duffing equations and a strong nonlinear oscillator. These problems pose a challenge to any method due to its feature of being sensitive dependent on initial conditions and some of them for their eventual chaotic behavior under certain range of parameters. The methodology herein presented is systematic and simple in its statement. It is very efficient in computing time. Neither divergence nor numerical damping are present in this examples nor others studied by the authors. It may be seen that rather large time steps were used and long time of experiments without occurrence of numerical problems. Finally, the availability of an analytical solution may be useful to perform qualitative analysis, which lead to the study of bifurcations and chaos.

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