Mecánica Computacional Vol. XXI, pp. 1263–1281
S. R. Idelsohn, V. E. Sonzogni and A. Cardona (Eds.)
Santa Fe–Paraná, Argentina, October 2002

ON THE RELAXED CONTINUITY APPROACH FOR THE SELF-REGULAR TRACTION-BIE

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Key words: Self-regular BEM, Relaxed continuity hypothesis, Traction-BIE, Elasticity 2D.

Abstract. *The 'relaxed continuity' hypothesis adopted on the self-regular traction-BIE is investigated for bidimensional problems. The self-regular traction-BIE, a fully regular equation, is derived from Somigliana stress identity, which contains hypersingular integrals. Due to the presence of hypersingular integrals the displacement field is required to achieve $C^{1,\alpha}$ Hölder continuity. This condition is not met by the use of standard conforming elements, based on C^0 interpolation functions, which only provide a piecewise $C^{1,\alpha}$ continuity. Thus, a relaxed continuity hypothesis is adopted, allowing the displacement field to be $C^{1,\alpha}$ piecewise continuous at the vicinity of the source point. The self-regular traction-BIE makes use of the displacement tangential derivatives, which are not part of the original BIE. The tangential derivatives are obtained from the derivative of the element interpolation functions. Therefore, two possible sources of error, which are the discontinuity of the displacement gradients at inter-element nodes and the approximation of the displacement tangential derivatives, are introduced. In order to establish the dominant error, non-conforming elements are implemented since they satisfy the continuity requirement at each collocation point. Standard Gaussian integration scheme is applied in the evaluation of all integrals involved. Quadratic, cubic and quartic isoparametric boundary elements are employed. Some numerical results are presented comparing the accuracy of conforming and non-conforming elements on the self-regular traction-BIE and highlighting the dominant error.*

1 INTRODUCTION

The boundary integral equation (BIE) has been shown to provide a convenient formulation to the analyses of many physical problems and to be a robust alternative to domain methods, such as the finite element method. Nevertheless, the search for an efficient and accurate approach to compute the singular integral has been a daunting task in the development of the boundary element method (BEM), since its first applications. Much research effort has been devoted to deal with singular integrals, leading to a wide variety of BEM algorithms with varying degrees of success and computational burdens. Among the devised algorithms, the ones based on the regularized form of the BIE are perhaps the most suitable approaches on the boundary element method.

Regularization procedures employed to reduce the order of singularity can be performed either locally or globally. Some examples of local regularization can be found in Ref. 1-4, where either a coordinate transformation or a subtraction of singularity was employed on the singular element to reduce the order of singularity of the integrands. In these cases, after regularization of the integrand, the standard numerical procedures used in the BEM can be applied.

The so-called self-regular formulation is a global regularization technique. In the self-regular formulations the original singular BIE is rewritten so that the integrals are self-regularized at those points where the integrals would be singular in the standard formulation, leaving at most weakly singular integrals over the entire domain. Therefore, the evaluation of the singular integrals in the Cauchy principal value (CPV) and/or Hadamard finite part (HFP) sense, as usually employed in the classical BEM can be avoided in the regularized algorithms. As pointed out by Rudolphi [5], the variables involved on the self-regular BEM are the same of the classical BEM. Moreover, the meaning of the original BIE is not changed due to regularization. The self-regular BIE have been spreading in boundary element community and applied to various fields, such as elasticity [6-12], potential theory [5,10,12-14], fracture mechanics [15-17], acoustics [18], thermoelasticity [19], elastodynamics [20], and so forth.

However, due to the presence of singular kernels on the BIE, a careful attention has to be paid to the smoothness of the density function. The smoothness requirement of the density functions for the self-regular BIE is the same of the respective hypersingular or strongly singular BIE.[21] In the primary BIE for elasticity problems (displacement-BIE) the displacement field should be $C^{0,\alpha}$ Hölder continuous. This continuity requirement is met by the use of standard conforming elements. Nevertheless, when dealing with the gradient-based BIE, i.e., traction-BIE for elasticity problems, the smoothness requirement is more stringent. The gradient-based BIE is derived from the differentiation of the primary BIE resulting on an integral equation containing kernels of higher order singularities (hypersingular and strongly singular kernels). In this case, for elasticity problems, the displacement field should be $C^{1,\alpha}$ Hölder continuous. Discretization of the boundary into standard conforming elements leads to a loss of the $C^{1,\alpha}$ continuity requirement of the displacement field at inter-element nodes.[21] Thus, *a priori* only boundary elements that ensure $C^{1,\alpha}$ continuity at each collocation point can be applied on the discretization of the referred BIE. In view of the smoothness requirement for the gradient-based BIE, four main approaches are adopted in the boundary element community.

The first approach is to use conforming boundary elements based on C^1 interpolation functions. These types of elements include both the Overhauser and the Hermite elements. When using the Overhauser elements,[22,23] the continuity of the first derivative of the variable field is implicitly enforced, without introducing extra variables to the original formulation. The continuity is achieved by employing information from the adjoining nodes on either side of the element. However, the shape functions of the Overhauser elements are more complex and not yet well developed for three-dimensional problems. In addition, the Overhauser element presents some disadvantages such as its inability to model discontinuities in the geometry. On the other hand, the use of Hermite elements[24] introduces new variables to the problem as the continuity is achieved by incorporating nodal tangential derivatives into the shape functions. The extra variables are computed through the use of the tangential derivative integral equation into the formulation. Thus, the computational cost of this approach is higher and its implementation can become someway cumbersome.

The second approach makes use of standard non-conforming elements.[5,19] These elements preserve the $C^{1,\alpha}$ continuity since all collocation point is placed at the interior of the elements. Nevertheless, quasi-singularities may appear, as the nodes have to be put very close to the end of the elements, leading to a bad conditioned system of equations. Additionally, the use of such elements implies on a higher system of equations to be solved as one extra node is created for each node at the intersection between two elements. Moreover, this approach does not guarantee the field variable to be single valued at inter-element nodes in spite of continuous solutions reported in the literature.

The third alternative is to adopt a variational approach. In this case $C^{1,\alpha}$ continuity is enforced at inter-element nodes and extra unknowns written as Lagrange multipliers are introduced. Extra subsidiary constraining equations are written for the discontinuities of displacement derivatives at inter-element nodes and this discontinuity is enforced to be zero. The smoothness requirement is therefore satisfied in a variational sense. The disadvantage of this approach is that the number of degrees of freedom increases for a given discretization. The variational approach for the self-regular traction-BIE presented in details by Jorge *et al.*[25]

The fourth approach relies on the relaxation of the smoothness requirement for the field variable. When adopting the ‘relaxed continuity’ hypothesis, standard conforming elements are used and it is allowed to collocate at inter-element nodes where only piece-wise $C^{1,\alpha}$ continuity is provided. Several authors have attempted to relax the continuity requirements in different fields.[6,8,10,11,13,14,17,18] In spite of good numerical results obtained by these authors, Martin and Rizzo[21] claim that the ‘relaxed continuity’ approach cannot be theoretically justified. Based on the numerical results achieved by Cruse and his co-workers[6,7,9] and the theoretical smoothness requirement, Martin *et al.*[26] renewed the discussion about the validation of the ‘relaxed continuity’ hypothesis. They pointed out that one of the possibilities of relaxing the continuity requirement is to assume sufficient smoothness to derive the self-regular BIE and then relaxing the smoothness requirement on the discretization of the integral equation. However, Martin *et* *al.*[26] state that since this approach is based on inconsistent reasoning, convergence and results accuracy cannot be assured. In the light of these discussions, Liu and Rudolphi[10] ask for a convergence study or a counter-example showing divergence in order to validate the 'relaxed continuity' approach.

This paper gives a contribution to the study of the validation of the 'relaxed continuity' hypothesis. The aim is to highlight the main source of error introduced on BEM solution when using the self-regular traction-BIE with conforming elements. There are two possible sources of error introduced by the self-regular traction-BIE, which do not occur if the standard BEM or the self-regular displacement BIE is applied. The regularizing term on the self-regular traction-BIE contains the displacement tangential derivatives that are not part of the original BIE. On the BEM algorithm, the tangential derivatives are locally evaluated through the exact differentiation of the element interpolation functions. As a result, the approximation of the tangential derivative is of one degree less than the approximation of the boundary variables[14], representing therefore one possible source of error. Indeed, several authors pointed out the evaluation of the tangential derivatives as the main source of error on the gradient-based formulations[11,14]. Their claim is based on better results achieved with higher order interpolation functions.

Another possible source of error on the self-regular traction-BIE with conforming elements is the assumption of the 'relaxed continuity' interpretation. If this hypothesis is adopted on the referred algorithm, the displacement gradients, which are part of the regularizing term, are not single-valued at each collocation point as derived analytically. Different values for the displacement gradient are used depending upon the source point location. Many authors believe that the use of the 'relaxed continuity' approach poses no significant errors on BEM solution.[10,11] Nevertheless, poor results recently reported by Cruz[27], Ribeiro *et al.*[28] and Ribeiro *et al*[29], specially when using quadratic elements, motivated a numerical study on the 'relaxed continuity' hypothesis.

Non-conforming elements are implemented herein in order to split these two sources of error, since for conforming elements both happen together. When non-conforming elements are employed, the displacement tangential derivatives are approximated through the derivatives of the element interpolation function, just like done for conforming elements, whereas the displacement gradients are single-valued at each collocation point, unlike for conforming elements.

The following sections present a brief review on the self-regular formulations and some features on BEM implementation. Numerical results for the self-regular traction-BIE with conforming and non-conforming quadratic, cubic and quartic elements are presented and compared to whether the exact solution or finite element method solution. The results are also compared to the self-regular displacement BEM solutions with conforming elements of the same order. The main source of error on the self-regular traction-BIE with conforming elements is pointed out.

2 SELF-REGULAR BIE

The general idea behind the regularization of the displacement-BIE and the traction-BIE are pointed out in this section.

2.1 Self-regular displacement-BIE

The well-known Somigliana displacement identity (SDI) (Eq.1) is and integral representation of the displacement at an interior point p in terms of the boundary displacements and tractions.

$$u_j(p) = -\int_S u_i(Q) T_{ji}(p,Q)\, dS(Q) + \int_S t_i(Q) U_{ji}(p,Q)\, dS(Q) \tag{1}$$

where Q is the integration point (boundary point), $u_i(Q)$ and $t_i(Q)$ are the displacement and traction fields, respectively, and $U_{ji}(p,Q)$ and $T_{ji}(p,Q)$ are the fundamental solutions.

When the source point moves from an interior point p to a boundary point P, the integrals in Eq.1 become strongly singular and weakly singular, respectively. Eq.1 is bounded if the displacement field is $C^{0,\alpha}$ Hölder continuos at the source point. This smoothness requirement implies that the first integral in Eq.1 should be analyzed in the Cauchy principal value (CPV) sense, as usually performed in the classical BEM. Nevertheless, if a simple solution corresponding to a rigid body motion is applied to Eq.1, a self-regularized form of the SDI is obtained, avoiding the evaluation of the first integral in the CPV sense. The following represents the self-regular SDI

$$u_j(p) - u_j(P) = -\int_S [u_i(Q) - u_i(P)] T_{ji}(p,Q)\, dS(Q) + \int_S t_i(Q) U_{ji}(p,Q)\, dS(Q) \tag{2}$$

Even though the regularization process is not "complete", Eq.2 presents a smooth transition of the displacements from interior to boundary points, provided the smoothness requirement is satisfied. The second integral in Eq.2, which was not regularized, is an improper integral when $p \to P$ and it is known to be bounded. The limit as $p \to P$ can be taken in Eq.2, given rise to the self-regular displacement-BIE.

$$0 = -\int_S [u_i(Q) - u_i(P)] T_{ji}(P,Q)\, dS(Q) + \int_S t_i(Q) U_{ji}(P,Q)\, dS(Q) \tag{3}$$

2.2 Self-regular traction-BIE

The self-regular traction-BIE is derived from the Somigliana stress identity (SSI) (Eq.4).

$$\sigma_{ij}(p) = -\int_S u_k(Q) S_{kij}(p,Q)\, dS(Q) + \int_S t_k(Q) D_{kij}(p,Q)\, dS(Q) \tag{4}$$

where the kernels $S_{kij}(p,Q)$ and $D_{kij}(p,Q)$ are the fundamental solutions and the densities $u_k(Q)$ and $t_k(Q)$ are the displacement and traction field.

Eq.4 contains a hypersingular and a strongly singular kernel in the limit as the source point moves from an interior point p to a boundary point P, ($p \rightarrow P$). Hypersingular integrals are known to exist when the density function is $C^{1,\alpha}$ Hölder continuous at the source point[21]. In this case, the displacement first derivatives must be continuous in the Hölder sense.

The most usual approach to deal with strongly singular and hypersingular integrals is to make use of Hadamard Finite Part (HFP) and/or CPV to derive a boundary integral equation for the Somigliana stress identity. An alternative and appealing approach is to use a simple solution to obtain a self-regular form of the SSI to derive the BIE. The self-regular SSI is obtained by subtracting and adding back a simple solution corresponding to a state of constant stress in the body that is equal to the boundary stress at a surface point P, leading to the following expression[8]

$$\sigma_{ij}(p) = \sigma_{ij}(P) - \int_S \left[u_k(Q) - u_k^L(Q)\right] S_{kij}(p,Q) dS(Q) + \int_S \left[t_k(Q) - t_k^L(Q)\right] D_{kij}(p,Q) dS(Q) \tag{5}$$

where $u_k^L(Q)$ and $t_k^L(Q)$ are the linear state of displacements and tractions associated with the boundary stress at P, and are given by

$$\begin{aligned} u_k^L(Q) &= u_k(P) + u_{k,m}(P)\left[x_m(Q) - x_m(P)\right] \\ t_k^L(Q) &= \sigma_{km}(P) n_m(Q) \end{aligned} \tag{6}$$

The coefficient $u_{k,m}(P)$ on Eq.6 represents the displacement gradients that are not part of the original identity.

Eq.5 is regular for all interior point limits to the boundary, including limits to the boundary at corners where continuity requirement $u(Q) \in C^{1,\alpha}$ is satisfied. Taking the limit as $p \rightarrow P$ a fully regular BIE that is valid for all boundary points is obtained. This equation is termed the self-regular traction-BIE and is given by the following

$$0 = -\int_S \left[u_k(Q) - u_k^L(Q)\right] S_{kij}(P,Q) dS(Q) + \int_S \left[t_k(Q) - t_k^L(Q)\right] D_{kij}(P,Q) dS(Q) \tag{7}$$

Contrary to the standard hypersingular BIE, Eq.7 does not require its integrands to be evaluated in the HFP and/or CPV sense.

3 SELF-REGULAR BEM ALGORITHMS

The main features of the self-regular BEM algorithms implemented in this work are described in this section.

3.1 Self-regular displacement BEM

In the discretized form of the self-regular displacement-BIE (Eq.3), only the first integral is regular, whereas the second integral remains weakly singular and needs no regularization. However, special care must be paid on the numerical evaluation of the weakly singular integral due to the logarithmic nature of the fundamental solution in 2D. In the current paper this integral is evaluated through a logarithm quadrature. A standard Gaussian quadrature is applied to numerically evaluate the first integral. The algorithms presented herein use quadratic, cubic and quartic boundary elements based on the standard isoparametric representations.

3.2 Self-regular traction-BEM

Although some features should be emphasized on the self-regular traction BEM, the basic assumptions are the same used for the self-regular displacement BEM. The BEM algorithm developed by Richardson and Cruse[11] for 2D elastostatics is taken as the basis of this study. A brief summary of this algorithm is presented herein. The algorithm requires an explicit representation of the displacement gradients as evaluated at the boundary. The displacement gradients are obtained for each boundary element in terms of the local displacement tangential derivatives and the local tractions. The displacement tangential derivatives are evaluated in terms of the intrinsic co-ordinate for each element in the following manner

$$\frac{du_k}{dS} \approx \frac{du_k(\xi)}{d\xi}\frac{d\xi}{dS} = \frac{1}{J(\xi)}\sum_{i=1}^{m} N_i'(\xi)u_k^i \qquad and \qquad N_i' = \frac{dN_i}{d\xi} \tag{8}$$

where m is the number of nodes per element, $J(\xi)$ is the Jacobian that is obtained in the usual manner from the isoparametric model of the element geometry, and $N_i'(\xi)$ is the derivative of the interpolation function. Therefore, the tangential derivatives are derived from polynomials of one degree less than the polynomials used to obtain the normal derivatives. Thus, the displacement gradient is obtained in a somewhat unbalanced manner.[14]

The displacement gradients at the source point are expressed in terms of nodal tractions and displacements through the mapping of the local tangential and normal derivatives of the displacement into the global coordinates.

$$u_{k,l}(P) \approx u_{k,l}(\xi^P) = A_{klr}(\xi^P)t_r(\xi^P) + B_{klr}(\xi^P)\sum_{i=1}^{m} N_i'(\xi^P)u_r^i \tag{9}$$

where $A_{klr}(\xi^P)$ and $B_{klr}(\xi^P)$ are the mapping functions[11]

The algorithm adopted in the current work uses standard conforming and non-conforming boundary elements. When non-conforming elements are used, the smoothness requirement for the displacement, which is continuous first derivative of the displacement field in the Hölder sense, is satisfied due to the fact that for such elements all collocation points are placed at intraelement nodes where $C^{1,\alpha}$ continuity is preserved. If conforming elements are used otherwise, such as done by Richardson and Cruse[11], the $C^{1,\alpha}$ Hölder continuity is not preserved. In order to overcome this problem some authors use a 'relaxed continuity' approach.[6,9,11,14,18] When this hypothesis is assumed, the displacement field is allowed to be $C^{1,\alpha}$ continuous only at the vicinity of the source point, so that conforming elements can be employed. Richardson *et al.*[9] stated that the BEM algorithm matches the analytical regularity condition required by the bounded BIE even though the $C^{1,\alpha}$ continuity of the displacement is not met.

If the 'relaxed continuity' hypothesis is assumed, the regularizing terms, which involve the displacement gradients, are no longer single-valued at inter-element nodes and depends upon source point location, in the following manner

$$u_k^{L^I}(\xi)=\begin{cases} u_k(P)+u_{k,m}(P)\left[x_m(\xi)-x_m(P)\right] & \text{for } P\in\Delta S_I \\ u_k(P)+\left(\dfrac{1}{2}\displaystyle\sum_{j=1}^{2}u_{k,m}(P_j)\right)\left[x_m(\xi)-x_m(P)\right] & \text{for } P\notin\Delta S_I \end{cases} \tag{10}$$

The average nodal value of the gradient evaluated based on the elements sharing the collocation point is used for the integrals in all elements, unless the element to be integrated contains the collocation point. In this case, the element-based gradient values employed are locally evaluated based on the interpolation scheme of this element. Using the element-based regularization (Eq.6), the resulting self-regular traction-BEM is obtained.

$$\begin{aligned} 0=&-\sum_{I=1}^{M}\int_{\Delta S_I}\left[u_k(\xi)-u_k^{L^I}(\xi)\right]S_{kij}(P,\xi)J(\xi)n_i(P)dS(\xi) \\ &+\sum_{I=1}^{M}\int_{\Delta S_I}\left[t_k(\xi)-t_k^{L^I}(\xi)\right]D_{kij}(P,\xi)J(\xi)n_i(P)dS(\xi) \end{aligned} \tag{11}$$

where M is the number of boundary elements and $n_i(P)$ is the surface normal at the collocation point.

The algorithm adopted herein allows for discontinuities in the surface normal and/or boundary traction, when conforming elements are employed, using single nodes at corners. Different values for the traction can be assigned to the elements sharing the corner, but if both tractions are unknowns, only an 'average' result is given. All integrals involved are evaluated through a standard Gaussian integration scheme, and quadratic isoparametric conforming and non-conforming boundary elements are used.

4 NUMERICAL EXAMPLES

Two bidimensional elastostatics problems are analyzed using the self-regular traction-BEM. Results for several meshes using conforming and non-conforming quadratic, cubic and quartic elements are obtained and compared to the exact solution or FEM solution. Also, BEM solutions from the self-regular displacement-BEM with conforming elements are taken as a comparison basis. A twelve-point Gaussian integration is employed in both examples. The problems are taken to be plane stress. The material constants are Poisson's ratio $\nu=0.3$ and Young's Modulus $E=205010$ units.

4.1 Cantilever beam

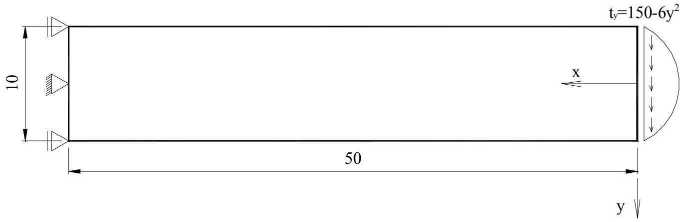

Figure 1: Geometry and Boundary Conditions of the Cantilever Beam

The first example involves the analysis of a cantilever beam subjected to a parabolic shear load at its end, equivalent to a concentrated load of 1000 units. The geometry and boundary conditions are shown in Fig.1. The coarsest mesh for each of the three elemental interpolation functions is constructed with twelve elements of equal size, where each vertical face is modeled with one element and each horizontal face is modeled with five elements.

In subsequent mesh refinements each element from the previous mesh is subdivided into two elements of equal size. BEM solutions for displacements throughout the boundary and normal traction along the constrained face for several meshes using quadratic, cubic and quartic interpolations are compared to the exact solutions to determine the maximum relative error. The regions where the principle of Saint-Venant is not valid are excluded in the evaluation of the maximum relative error. The results for the magnitude of the error are plotted in Fig.2-4 for the self-regular traction-BIE with both conforming and non-conforming quadratic (Fig.2), cubic (Fig.3) and quartic (Fig.4) boundary elements.

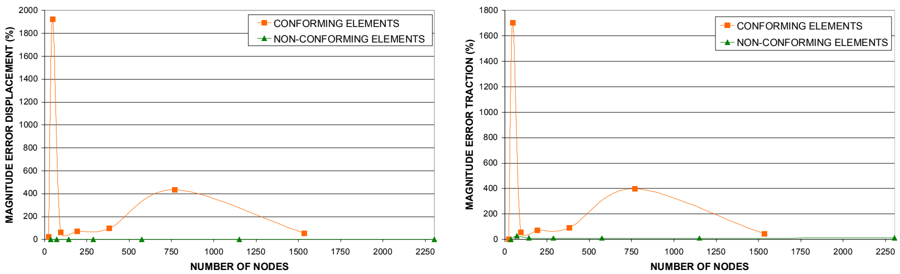

Figure 2: Cantilever Beam: magnitude of the error in the self-regular traction-BEM with quadratic elements

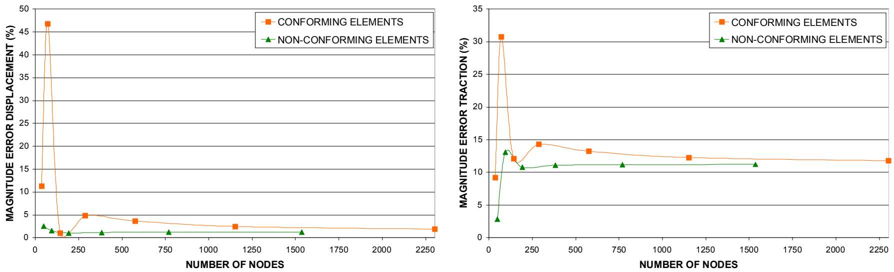

Figure 3: Cantilever Beam: magnitude of the error in the self-regular traction-BEM with cubic elements

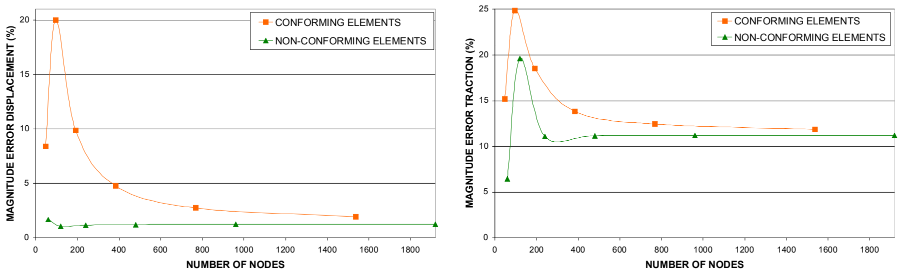

Figure 4: Cantilever Beam: magnitude of the error in the self-regular traction-BEM with quartic elements

If a maximum error of three per cent in boundary displacements is assumed as a convergence criterion, the BEM results using the self-regular traction-BEM with conforming elements show that the problem requires more than 768 quadratic elements, 384 cubic elements or 192 quartic elements, whereas using non-conforming elements only 12 quadratic, cubic or quartic elements are necessary. An improvement in results accuracy can be noticed when non-conforming elements are used instead of conforming elements on the self-regular traction-BEM. Furthermore, the oscillatory convergence behavior of the displacement results obtained with the use of conforming elements is not noticed when non-conforming elements are employed. In addition, it can be noticed from Fig.5-7 that the results for the self-regular traction-BEM with non-conforming elements (SRTB-NC) show the same level of accuracy as the results for the self-regular displacement-BEM with conforming elements (SRDB).

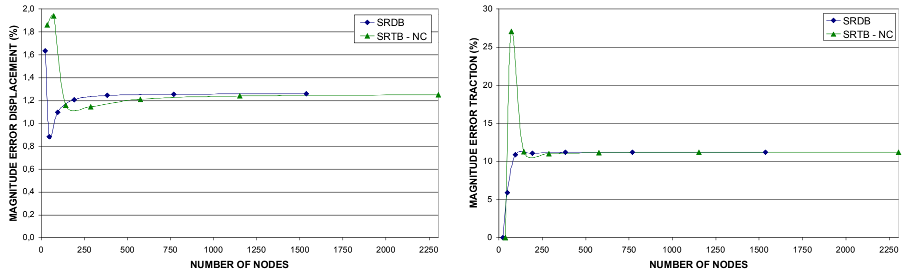

Figure 5: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Quadratic elements

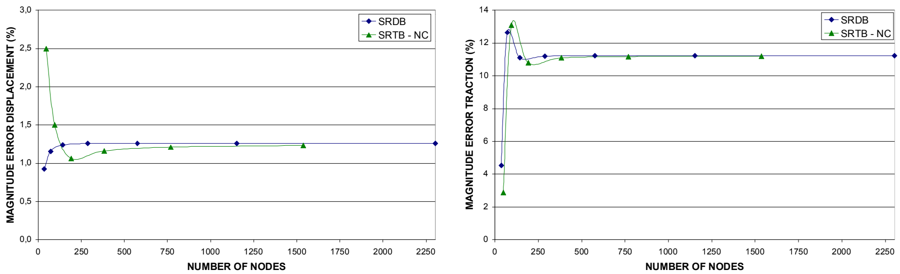

Figure 6: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Cubic elements

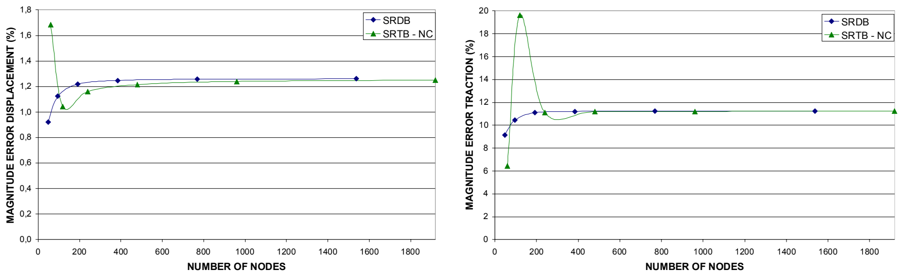

Figure 7: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Quartic elements

The improvement in the accuracy of the results achieved by the use of quadratic non-conforming elements on the self-regular traction-BIE instead of conforming elements of the same order is higher than for cubic elements, which in its turn is higher than for quartic elements (Fig.2-4).

It is our contention that the different improvement rates in the results accuracy obtained for each of the three elements are in someway related to the proportion of inter-element to intraelement nodes. In a closed boundary, the use of conforming quadratic elements implies that there is one intraelement node to each node at the junction between two elements. This means that when adopting the 'relaxed continuity' hypothesis, for each collocation point where the displacement tangential derivative is single-valued, there is one collocation point where different values for the displacement gradient are assigned, according to the element to be integrated. If cubic elements are employed the proportion is 1:2 and for quartic elements the proportion is 1:3. Thus, the error introduced by the assumption of the 'relaxed continuity' hypothesis is more critical for quadratic boundary elements than for higher order elements.

From this example, the error results in the self-regular traction-BIE formulation seems to be more dependent on the discontinuity of the displacement gradients at inter-element nodes than on the interpolation of displacement tangential derivative. Apparently, the assumption of different values for the displacement gradient, according to its location (Eq.10), has greater influence on the solution accuracy than the approximate evaluation of the displacement tangential derivative that is based on each element interpolation function.

Previous work with the self-regular gradient-based BIE formulations for 2-D problems[11,14] pointed out to the tangential derivative interpolation as the dominant error source, for various degrees of the interpolating functions. The current work shows otherwise that the tangential derivative interpolation might not be the dominant error source for all cases. Indeed, in this first example the dominant error source on the self-regular traction-BIE is shown to be the discontinuity of the gradient at the inter-element nodes.

4.2 Rectangular domain subjected to a concentrated load

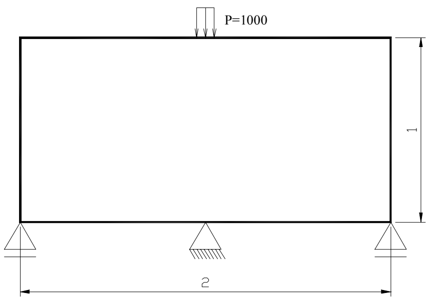

Figure 8: Geometry and boundary conditions to the rectangular domain subjected to a "concentrated load"

A rectangular domain subjected to a "concentrated" load is analyzed. The geometry and boundary conditions are shown in Fig.8. The "concentrated" load of 1000 units is distributed along a small element which size is equal to ten per cent the size of the adjacent elements.

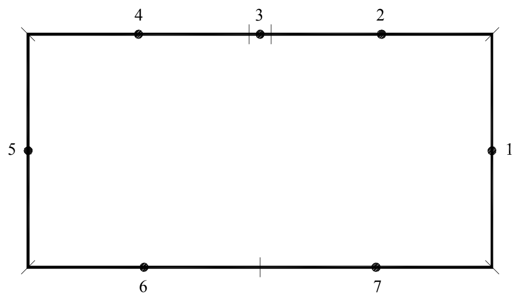

Fig.9: Problem 2: Coarsest mesh with quadratic elements

The coarsest mesh for each of the three types of elements is constructed with seven elements, where each vertical face is modeled with one element, the bottom face is modeled with two elements of the same size and the top face is modeled with three elements: two elements of the same size and one small element, as shown in Fig.9, where the mesh is constructed with quadratic elements and only the number of the elements are shown. Starting from the coarsest mesh, several refined meshes were generated. Each mesh has been created by dividing the elements on the vertical and bottom faces of each previous mesh into two elements of equal size. The refinement at the top face is performed in such a way that the face is modeled with one element more than the bottom face and the proportion of the small element to the adjacent elements remains the same (10%).

BEM solutions for vertical displacement at the midpoint of the vertical face and normal tractions at the bottom face are obtained for several meshes and compared to FEM solutions to determine the relative error. The self-regular algorithms presented herein are used with quadratic, cubic and quartic elements. Results for the maximum relative error for the self-regular traction-BIE with conforming and non-conforming elements are presented in Fig.10-12.

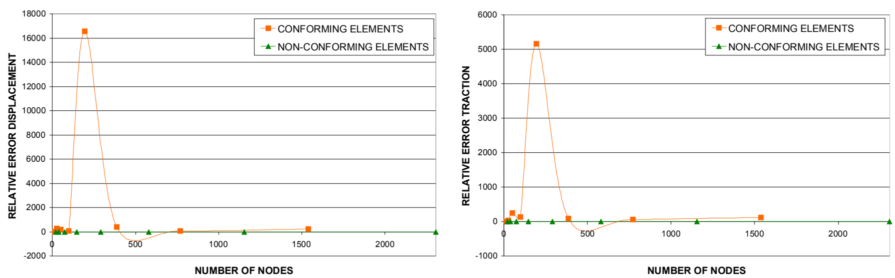

Figure 10: Relative error in the self-regular traction-BEM with quadratic elements

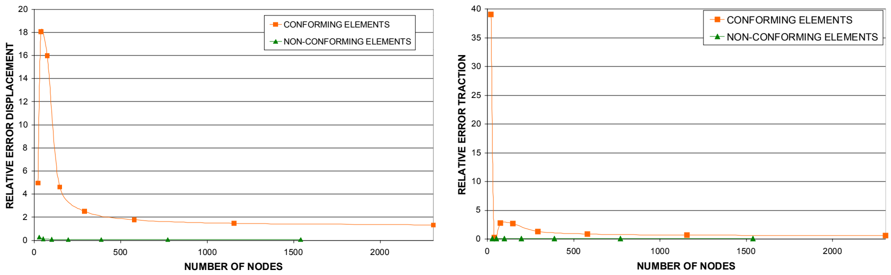

Figure 11: Relative error in the self-regular traction-BEM with cubic elements

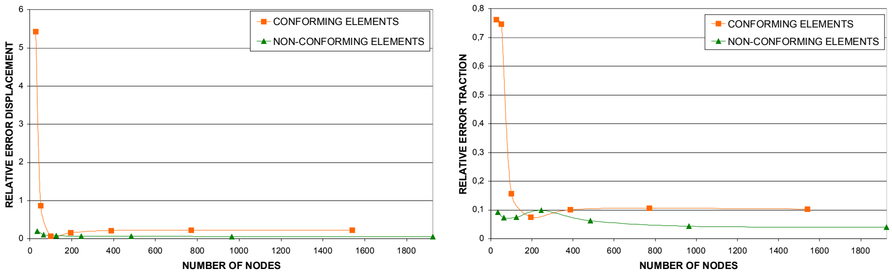

Figure 12: Relative error in the self-regular traction-BEM with quartic elements

Again, it can be noticed that the accuracy of the results for the self-regular traction-BIE is improved when non-conforming elements are implemented. Also, the improvement rate is higher for quadratic elements than for cubic and quartic elements. Unlike the first example, the oscillatory convergence behavior of the displacement results is only observed for quadratic and cubic elements. The use of non-conforming quartic elements instead of conforming elements of the same order presents almost no gain in the results accuracy. The results from the self-regular traction-BIE with non-conforming elements are almost on the same level of accuracy as the results from the self-regular displacement-BIE, even though in some cases this comparison is not as good as for the first example (Fig.13-15).

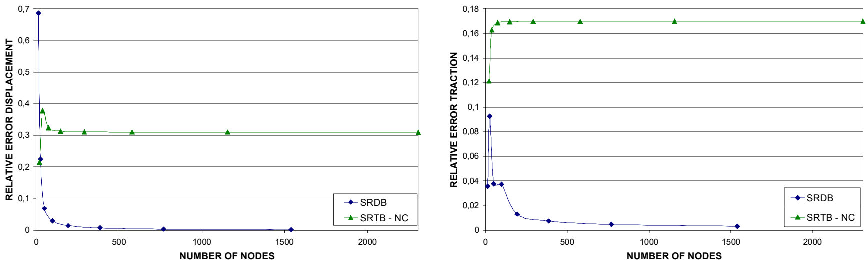

Figure 13: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Quadratic elements

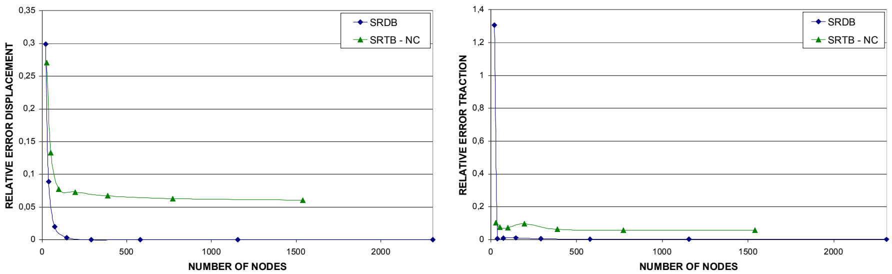

Figure 14: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Cubic elements

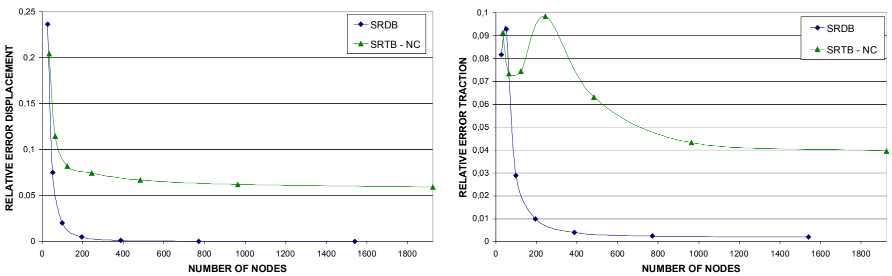

Figure 15: Non-conforming self-regular traction-BEM x self-regular displacement-BEM: Quartic elements

From this example, it seems that the assumption of the ‘relaxed continuity’ hypothesis has indeed a great influence of the accuracy of the results from the self-regular traction-BIE with conforming elements. This influence is not so pronounced for quartic elements since, as previously explained, when these elements are used, for each inter-element node, where there is a discontinuity on the displacement gradient, there are three intraelement nodes where the gradients are single-valued. On the other hand, the discontinuity of the displacement gradients has greater influence in results accuracy when quadratic and cubic elements are used, since the proportion of inter-element to intralement nodes is lower than for quartic elements. These results therefore strengthen the conclusions drawn to the first example.

5 CONCLUSIONS

The self-regular formulations for bidimensional elastostatics are reviewed, and the main features of their implementation are discussed. A study regarding the possible errors introduced on the discretization of the self-regular traction-BIE with conforming elements has been presented. The approximation of the displacement tangential derivative and discontinuity of the displacement gradients at the collocation point, which is the essence of the 'relaxed continuity' hypothesis, are pointed out as possible sources of error introduced by the self-regular traction-BIE. When the 'relaxed continuity' hypothesis is adopted on the implementation of the currrent algorithm, the displacement gradients may be considered as an average of the nodal gradients or as the gradient locally evaluated based on a specific element, depending on the element to be integrated. If conforming boundary elements are used on the self-regular traction-BIE both errors can be important on the evaluation of displacements and tractions at the boundary. The implementation of non-conforming elements allows these sources of error to be split, highlighting the most important one.

Two examples are analyzed and numerical results are obtained for the self-regular traction-BEM with conforming and non-conforming quadratic, cubic an quartic boundary elements. BEM solutions from the self-regular displacement-BIE are also presented. The first example consists of a cantilever beam subjected to a parabolic shear load, for which the analytical solution is available. The second example in its turn involves a rectangular domain subjected to a concentrated load, and its results are compared to the FEM solutions.

In both examples a significant gain in solution accuracy can be noticed through the use of non-conforming quadratic elements on the self-regular traction-BIE instead of conforming quadratic elements. The use of cubic and quartic non-conforming elements does not present a such high gain in results accuracy when compared to conforming elements of the same order. Nevertheless, when non-conforming elements are implemented on the self-regular traction-BIE, the oscillatory convergence behavior obtained for conforming elements is not noticed in most of the cases. Moreover, the results form the self-regular traction-BIE with non-conforming elements achieved the same level of accuracy as the BEM solutions from the self-regular displacement-BIE.

The interpolation of the tangential derivative does not seem to be the dominant error source for the self-regular traction-BIE, as previously stated. This work on the contrary, showed that the discontinuity of the gradient at the inter-element nodes appears to be the dominant error source.

The results shown here strengthen the conclusions recently presented by Ribeiro *et al.*[30]. The fact that different values are assign to the regularizing gradient according to the element to be integrated seems to introduce high errors on the results using the self-regular traction-BIE with conforming elements. This kind of error is not observed when the source point is placed at the interior of the element, where the smoothness requirement is preserved. The analysis using quadratic elements is the most critical. A plausible explanation for better results achieved with higher order elements is that when using quadratic elements for each equation related to an inter-element node there is one equation related to an intralement node, which represents a proportion of 1:1, for cubic and quartic elements this proportion is higher (1:2 and 1:3 respectively).

Although the use of standard conforming elements is more appealing and easier than the use of non-conforming elements or elements based on C^1 interpolation functions, from the results obtained so far, it seems that BEM solutions from the self-regular traction-BIE using this kind of element are not reliable. It appears that, as pointed by Martin and Rizzo[21], the smoothness requirement on the self-regular traction-BIE should be satisfied in order to guarantee results accuracy. A more detailed numerical study is being conducted in order to achieve a final conclusion regarding the reliability of the self-regular traction-BIE with conforming elements.

6 ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support received from CAPES-Coordenação de Aperfeiçoamento de Pessoal de Nível Superior and CNPq-Conselho Nacional de Desenvolvimento Científico e Tecnológico, both Brazilian Government Agencies.

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